STAT253/317 Lecture 4: 4.4 Limiting Distribution I

Stationary Distribution

Define $\pi_i^{(n)} = \mathrm{P}(X_n = i)$, $i \in \mathfrak{X}$ to be the marginal distribution of X_n, $n = 1, 2, \ldots$, and let $\pi^{(n)}$ be the row vector

$$\pi^{(n)} = (\pi_0^{(n)}, \pi_1^{(n)}, \pi_2^{(n)}, \ldots),$$

From Chapman-Kolmogrov Equation, we know that

$$\pi^{(n)} = \pi^{(n-1)}\mathbb{P} \quad \text{i.e.} \quad \pi_j^{(n)} = \sum\nolimits_{i\in\mathfrak{X}} \pi_i^{(n-1)} P_{ij} \text{ for all } j \in \mathfrak{X},$$

If π is a distribution on $\mathfrak{X}$ satisfying

$$\pi\mathbb{P} = \pi \quad \text{i.e.} \quad \pi_j = \sum\nolimits_{i\in\mathfrak{X}} \pi_i P_{ij} \text{ for all } j \in \mathfrak{X},$$

then $\pi^{(0)} = \pi$ implies $\pi^{(n)} = \pi$ for all n.

We say π is a **stationary distribution** of the Markov chain.

Example 1: 2-state Markov Chain

$$\mathfrak{X} = \{0, 1\}, \qquad \mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

$$\pi\mathbb{P} = \pi \Rightarrow \begin{cases} \pi_0 &= (1-\alpha)\pi_0 + \beta\pi_1 \\ \pi_1 &= \alpha\pi_0 + (1-\beta)\pi_1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha\pi_0 &= \beta\pi_1 \\ \beta\pi_1 &= \alpha\pi_0 \end{cases}$$

Need one more constraint: $\pi_0 + \pi_1 = 1$

$$\Rightarrow \pi = (\pi_0, \pi_1) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$$

Example 2: Ehrenfest Diffusion Model

$$P_{ij} = \begin{cases} \dfrac{i}{2a} & \text{if } j = i-1 \\ \dfrac{2a-i}{2a} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\pi_0 &= \pi_1 P_{10} = \tfrac{1}{2a}\pi_1 \Rightarrow \pi_1 = 2a\pi_0 = \tbinom{2a}{1}\pi_0 \\
\pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = \pi_0 + \tfrac{2}{2a}\pi_2 \Rightarrow \pi_2 = \tfrac{2a(2a-1)}{2}\pi_0 = \tbinom{2a}{2}\pi_0 \\
\pi_2 &= \pi_1 P_{12} + \pi_3 P_{32} = \tfrac{2a-1}{2a}\pi_1 + \tfrac{3}{2a}\pi_3 \Rightarrow \pi_3 = \tfrac{2a(2a-1)(2a-2)}{6}\pi_0 = \tbinom{2a}{3}\pi_0 \\
&\vdots \qquad \ldots
\end{aligned}$$

In general,

$$\pi_i = \binom{2a}{i}\left(\frac{1}{2}\right)^{2a} \quad \text{for } i = 0, 1, 2, \ldots, 2a$$

Stationary Distribution May Not Be Unique

Consider a Markov chain with transition matrix $\mathbb{P}$ of the form

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \end{array} = \begin{pmatrix} \mathbb{P}_x & 0 \\ 0 & \mathbb{P}_y \end{pmatrix}$$

This Markov chain has 2 classes {0,1} and {2, 3, 4}, both closed and recurrent. So this Markov chain can be reduced to two sub-Markov chains, one with state space {0,1} and the other {2, 3, 4}. Their transition matrices are respectively $\mathbb{P}_x$ and $\mathbb{P}_y$.

Say $\pi_x = (\pi_0, \pi_1)$ and $\pi_y = (\pi_2, \pi_3, \pi_4)$ be respectively the stationary distributions of the two sub-Markov chains, i.e.,

$$\pi_x\mathbb{P}_x = \pi_x, \quad \pi_y\mathbb{P}_y = \pi_y$$

Verify that $\pi = (c\pi_0, c\pi_1, (1-c)\pi_2, (1-c)\pi_3, (1-c)\pi_3)$ is a stationary distribution of $\{X_n\}$ for any c between 0 and 1.

Not All Markov Chains Have a Stationary Distribution

For one-dimensional symmetric random walk, the transition probabilities are

$$P_{i,i+1} = P_{i,i-1} = 1/2$$

Try to solve for the stationary distribution

$$\pi_j = \sum\nolimits_{i\in\mathfrak{X}} \pi_i P_{ij} = \frac{1}{2}\pi_{j-1} + \frac{1}{2}\pi_{j+1}$$

π_j must be of the form

$$\pi_j = a + bj, \quad \text{for all integer } j \text{ and some constants } a, b$$

To make $\pi_j \geq 0$ for all integer j, $\Rightarrow b = 0$. Thus

$$\pi_j = a \quad \text{for all integer } j$$

Impossible to make $\sum_{j=-\infty}^{\infty} \pi_j = 1$.

Conclusion: 1-dim symmetric random walk does not have a stationary distribution.

Limiting Distribution

A Markov chain is said to have a **limiting distribution** if for all $i, j \in \mathfrak{X}$, $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exists, independent of the initial state X_0, and π_j's satisfy $\sum_{j\in\mathfrak{X}} \pi_j = 1$.

$$\text{i.e.,} \quad \lim_{n\to\infty} \mathbb{P}^n = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Limiting Distribution is a Stationary Distribution

The limiting distribution of a Markov chain is a stationary distribution of the Markov chain.

Proof *(not rigorous)*. By Chapman Kolmogorov Equation,

$$P_{ij}^{(n+1)} = \sum_{k \in \mathfrak{X}} P_{ik}^{(n)} P_{kj}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\pi_j = \lim_{n\to\infty} P_{ij}^{(n+1)} &= \lim_{n\to\infty} \sum_{k \in \mathfrak{X}} P_{ik}^{(n)} P_{kj} \\ &=^* \sum_{k \in \mathfrak{X}} \lim_{n\to\infty} P_{ik}^{(n)} P_{kj} \quad \text{(needs justification)} \\ &= \sum_{k \in \mathfrak{X}} \pi_k P_{kj}\end{aligned}$$

Thus the limiting distribution π_j's satisfies the equations $\pi_j = \sum_{k\in\mathfrak{X}} \pi_k P_{kj}$ for all $j \in \mathfrak{X}$ and is a stationary distribution.

See Karlin & Taylor (1975), Theorem 1.3 on p.85-86 for a rigorous proof.

Limiting Distribution is Unique

If a Markov chain has a limiting distribution π, then

$$\lim_{n\to\infty} \pi_j^{(n)} = \pi_j \text{ for all } j \in \mathfrak{X}, \text{ whatever } \pi^{(0)} \text{ is}$$

Proof *(not rigorous)*. Since

$$\pi_j^{(n)} = \sum_{k \in \mathfrak{X}} \pi_k^{(0)} P_{kj}^{(n)}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} \pi_j^{(n)} &= \lim_{n\to\infty} \sum_{k \in \mathfrak{X}} \pi_k^{(0)} P_{kj}^{(n)} \\ &=^* \sum_{k \in \mathfrak{X}} \pi_k^{(0)} \lim_{n\to\infty} P_{kj}^{(n)} \quad \text{(needs justification)} \\ &= \underbrace{\sum_{k \in \mathfrak{X}} \pi_k^{(0)}}_{=1} \pi_j = \pi_j\end{aligned}$$

i.e., if a limiting distribution exists, it is the unique stationary distribution.

Example: Two-State Markov Chain

$$\mathfrak{X} = \{0, 1\}, \qquad \mathbb{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array}$$

By induction, one can show that

$$\begin{aligned}\mathbb{P}^n &= \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n \\ \frac{\beta}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \end{pmatrix} \\ &\to \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix} \quad \text{as } n \to \infty\end{aligned}$$

The limiting distribution π is $(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$.

Not All Markov Chains Have Limiting Distributions I

Consider simple random walk on $\{0, 1, 2, 3, 4\}$ with absorbing boundary.

$$\mathbb{P} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Classes: {0}, {1,2,3}, {4}. One can show that

$$\mathbb{P}^n \to \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3/4 & 0 & 0 & 0 & 1/4 \\ 2 & 2/4 & 0 & 0 & 0 & 2/4 \\ 3 & 1/4 & 0 & 0 & 0 & 3/4 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array} \quad \text{as } n \to \infty$$

This Markov chain has no limiting distribution since $\lim_{n\to\infty} P_{ij}^n$ depends on the initial state i.

Not All Markov Chains Have Limiting Distributions II

For example, in the Ehrenfest diffusion model,

$$\mathbb{P}^{2n} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \ldots & 2a \\ \hline 0 & * & 0 & * & 0 & \ldots & * \\ 1 & 0 & * & 0 & * & \ldots & 0 \\ 2 & * & 0 & * & 0 & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2a & * & 0 & * & 0 & \ldots & * \end{array}$$

In fact, as $n \to \infty$,

$$P_{ij}^{2n} \to \begin{cases} 2\binom{2a}{j}(\frac{1}{2})^{2a} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases}$$

$$P_{ij}^{2n+1} \to \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 2\binom{2a}{j}(\frac{1}{2})^{2a} & \text{if } i+j \text{ is odd} \end{cases}$$

$$\mathbb{P}^{2n+1} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \ldots & 2a \\ \hline 0 & 0 & * & 0 & * & \ldots & 0 \\ 1 & * & 0 & * & 0 & \ldots & * \\ 2 & 0 & * & 0 & * & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2a & 0 & * & 0 & * & \ldots & 0 \end{array}$$

$\lim_{n\to\infty} P_{ij}^n$ doesn't exist for all $i, j \in \mathfrak{X}$

Periodicity

A state of a Markov chain is said to have **period** d if

$$P_{ii}^n = 0, \quad \text{whenever } n \text{ is not a multiple of } d$$

In other words, d is the *greatest common divisor* of all the n's such that

$$P_{ii}^n > 0$$

We say a state is **aperiodic** if $d = 1$, and **periodic** if $d > 1$.

For the Ehrenfest diffusion model, all states are of period $d = 2$.

Fact: Periodicity is a class property.
That is, all states in the same class have the same period.

For a proof, see Problem 2&3 on p.77 of Karlin & Taylor (1975).