

STAT253/317 Winter 2014 Lecture 10

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5.3 The Poisson Processes

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Outline:

- ▶ Interarrival times of events are i.i.d Exponential with rate λ
- ▶ Conditional Distribution of the Arrival Times
- ▶ Superposition & Thinning (Lecture 11)
- ▶ "Converse" of Superposition & Thinning (Lecture 11)

Arrival & Interarrival Times of Poisson Processes

Let

- S_n = Arrival time of the n -th event, $n = 1, 2, \dots$
- $T_1 = S_1$ = Time until the 1st event occurs
- $T_n = S_n - S_{n-1}$
= time elapsed between the $(n - 1)$ st and n -th event,
 $n = 2, 3, \dots$

Proposition 5.1

The interarrival times $T_1, T_2, \dots, T_k, \dots$, are i.i.d $\sim \text{Exp}(\lambda)$.

Consequently, as the distribution of the sum of n i.i.d $\text{Exp}(\lambda)$ is $\text{Gamma}(n, \lambda)$, the arrival time of the n th event is

$$S_n = \sum_{i=1}^n T_i \sim \text{Gamma}(n, \lambda)$$

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Proof of Proposition 5.1

$$\begin{aligned} & P(T_{n+1} > t | T_1 = t_1, T_2 = t_2, \dots, T_n = t_n) \\ &= P(0 \text{ event in } (s_n, s_n + t] | T_1 = t_1, T_2 = t_2, \dots, T_n = t_n) \\ & \quad \text{(where } s_n = t_1 + t_2 + \dots + t_n) \\ &= P(0 \text{ event in } (s_n, s_n + t]) \quad \text{(by indep increment)} \\ &= P(N(s_n + t) - N(s_n) = 0) \\ &= e^{-\lambda t} \quad \text{(stationary increment)} \end{aligned}$$

This shows that T_{n+1} is $\sim \text{Exp}(\lambda)$, and is independent of T_1, T_2, \dots, T_n .

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Definition 3 of the Poisson Process

A continuous-time stochastic process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

- (i) $N(0) = 0$,
- (ii) $N(t)$ counts the number of events that have occurred up to time t (i.e. it is a counting process).
- (iii) The times between events are independent and identically distributed with an $\text{Exponential}(\lambda)$ distribution.

We have seen how Definition 5.1 implies (i), (ii) and (iii) in Definition 3. The proof of the converse is omitted.

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5.3.5. Conditional Distribution of the Arrival Times

Uniform Distribution of Arrivals

Given $N(t) = 1$, then T_1 , the arrival time of the first event $\sim \text{Unif}(0, t)$

Proof.

$$\begin{aligned} P(T_1 \leq s | N(t) = 1) &= \frac{P(T_1 \leq s, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P(1 \text{ event in } (0, s], \text{ no events in } (s, t])}{P(N(t) = 1)} \\ &= \frac{(\lambda s e^{-\lambda s})(e^{-\lambda(t-s)})}{\lambda t e^{-\lambda t}} = \frac{s}{t}, s < t. \end{aligned}$$

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Theorem 5.2

Given $N(t) = k$,

$$(S_1, S_2, \dots, S_k) \sim (U_{(1)}, U_{(2)}, \dots, U_{(k)})$$

where $(U_{(1)}, \dots, U_{(k)})$ are the order statistics of $(U_1, \dots, U_k) \sim$ i.i.d Uniform $(0, t)$, i.e., the joint conditional density of S_1, S_2, \dots, S_k is

$$f(s_1, s_2, \dots, s_k | N(t) = k) = k! t^{-k}, \quad 0 < s_1 < s_2 < \dots < s_k$$

Proof. The event that $S_1 = s_1, S_2 = s_2, \dots, S_n = s_n, N(t) = n$ is equivalent to the event $T_1 = s_1, T_2 = s_2 - s_1, \dots, T_n = s_n - s_{n-1}, T_{n+1} > t - s_n$. Hence, by Proposition 5.1, we have the conditional joint density of S_1, \dots, S_n given $N(t) = n$ as follows:

$$\begin{aligned} f(s_1, \dots, s_n | n) &= \frac{f(s_1, \dots, s_n, n)}{P(N(t) = n)} \\ &= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \dots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t} (\lambda t)^n / n!} \\ &= n! t^{-n}, \quad 0 < s_1 < \dots < s_n < t \end{aligned}$$

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Example 5.21. Insurance claims comes according to a Poisson process $N(t)$ with rate λ . Let

- ▶ S_i = the time of the i th claims
- ▶ C_i = amount of the i th claims, i.i.d with mean μ , indep. of $\{N(t)\}$

Then the total discounted cost by time t at discount rate α is given by

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

Then

$$\begin{aligned} \mathbb{E}[D(t) | N(t)] &= \mathbb{E} \left[\sum_{i=1}^{N(t)} C_i e^{-\alpha S_i} \middle| N(t) \right] \stackrel{(Thm 5.2)}{=} \mathbb{E} \left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_{(i)}} \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_i} \right] = \sum_{i=1}^{N(t)} \mathbb{E}[C_i] \mathbb{E} \left[e^{-\alpha U_i} \right] \\ &= N(t) \mu \int_0^t \frac{1}{t} e^{-\alpha x} dx = N(t) \frac{\mu}{\alpha t} (1 - e^{-\alpha t}) \end{aligned}$$

$$\text{Thus } \mathbb{E}[D(t)] = \mathbb{E}[N(t)] \frac{\mu}{\alpha t} (1 - e^{-\alpha t}) = \frac{\lambda \mu}{\alpha} (1 - e^{-\alpha t})$$

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