Generating Functions

For a non-negative-integer-valued random variable $T$, the generating function of $T$ is the expected value of $s^T$

$$G(s) = E[s^T] = \sum_{k=0}^{\infty} s^k P(T = k),$$

in which $s^T$ is defined as 0 if $T = \infty$.

- $G(s)$ is a power series converging absolutely for all $-1 < s < 1$.
- $G(1) = P(T < \infty) \begin{cases} 1 & \text{if } T \text{ is finite w/ prob. 1} \\ < 1 & \text{otherwise} \end{cases}$

Knowing $G(s) \iff$ Knowing $P(T = k)$ for all $k = 0, 1, 2, \ldots$

More Properties of Generating Functions

- $E[T] = \lim_{s \to 1^-} G'(s)$ if it exists because

  $$G'(s) = \frac{d}{ds} E[s^T] = E[Ts^{T-1}] = \sum_{k=1}^{\infty} s^{k-1} kP(T = k).$$

- $E[T(T-1)] = \lim_{s \to 1^-} G''(s)$ if it exists because

  $$G''(s) = E[T(T-1)s^{T-2}] = \sum_{k=2}^{\infty} s^{k-2} k(k-1)P(T = k).$$

- If $T$ and $U$ are independent non-negative-integer-valued random variables, with generating function $G_T(s)$ and $G_U(s)$ respectively, then the generating function of $T + U$ is

  $$G_{T+U}(s) = E[s^{T+U}] = E[s^T]E[s^U] = G_T(s)G_U(s)$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p, P_{i,i-1} = 1-p = q$, for $i = 1, 2, 3, \ldots$
- Only one class, irreducible
- For $i < j$, define

  $$N_{ij} = \min\{m > 0 : X_m = j | X_0 = i\}$$

  = time to reach state $j$ starting in state $i$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$

  By the Markov property, $N_{01}, N_{12}, \ldots, N_{n-1,n}$ are indep.

- Given $X_0 = i$

  $$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i + 1 \\ 1 + N_{i-1,i}^* + N_{i,i+1}^* & \text{if } X_1 = i - 1 \end{cases}$$  \hspace{1cm} (1)

  where $N_{i-1,i}^*, N_{i,i+1}^*$ are indep.
Generating Function of \(N_{i,i+1}\)

Let \(G_i(s)\) be the generating function of \(N_{i,i+1}\). From (??), and by the independence of \(N_{i-1,i}\) and \(N_{i,i+1}\), we get that

\[
G_i(s) = ps + qE[s^{1+N_{i-1,i}+N_{i,i+1}}] = ps + qG_{i-1}(s)G_i(s)
\]

So

\[
G_i(s) = \frac{ps}{1 - qG_{i-1}(s)}
\]

Since \(N_{01}\) is always 1, we have \(G_0(s) = s\). Using the iterating relation (??), we can find

\[
G_1(s) = \frac{ps}{1 - qG_0(s)} = \frac{ps}{1 - q^2} = ps \sum_{k=0}^{\infty} (qs^2)^k = \sum_{k=0}^{\infty} pq^k s^{2k+1}
\]

So \(P(N_{12} = n) = \begin{cases} pq^k & \text{if } n = 2k + 1 \text{ for } k = 0, 1, 2 \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}\)

Mean of \(N_{i,i+1}\)

Recall that \(G_i'(1) = E(N_{i,i+1})\). Let \(m_i = E(N_{i,i+1}) = G_i'(1)\). Then

\[
G_i'(s) = \frac{p(1 - qG_{i-1}(s)) + ps(qG_{i-1}(s) + qG_i'(s))}{(1 - qG_{i-1}(s))^2}
\]

Since \(N_{i,i+1} < \infty\), \(G_i(1) = 1\) for all \(i = 0, 1, \ldots, n - 1\). We have

\[
m_i = G_i'(1) = \frac{p + pqG_i'(1)}{1 - qG_i'(1)} = \frac{1 + qG_i'(1)}{p} = \frac{1}{p} + \frac{q}{p} m_{i-1}
\]

\[
= \frac{1}{p} \left[ 1 + \frac{q}{p} + \left( \frac{q}{p} \right)^2 + \ldots + \left( \frac{q}{p} \right)^{i-1} \right] + \left( \frac{q}{p} \right)^i m_0
\]

Since \(N_{01} = 1\), which implies \(m_0 = 1\).

\[
m_i = \begin{cases} \frac{1 - (q/p)}{p-q} + \left( \frac{q}{p} \right)^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}
\]

Mean of \(N_{0,n}\)

Recall that \(N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}\)

\[
E[N_{0n}] = m_0 + m_1 + \ldots + m_{n-1}
\]

\[
= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2} \left[ 1 - \left( \frac{q}{p} \right)^n \right] & \text{if } p \neq 0.5 \\ \frac{n^2}{p} & \text{if } p = 0.5 \end{cases}
\]

When

\[
p > 0.5 \quad E[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2} \left[ 1 - \left( \frac{q}{p} \right)^n \right] \quad \text{linear in } n
\]

\[
p = 0.5 \quad E[N_{0n}] = n^2 \quad \text{quadratic in } n
\]

\[
p < 0.5 \quad E[N_{0n}] = O\left( \frac{2pq}{(p-q)^2} \left( \frac{q}{p} \right)^n \right) \quad \text{exponential in } n
\]
Example: Symmetric Random Walk on \((-\infty, \infty)\)

State space = \{\ldots, -2, -1, 0, 1, 2, \ldots\}

\[ P_{i,i+1} = P_{i,i-1} = 1/2, \quad \text{for all integer } i \]

- 1 classes, recurrent, null-recurrent or positive-recurrent?
- For \(i, j\), define \(N_{ij} = \min\{m > 0 : X_m = j | X_0 = i\}\).
- Note \(N_{00} = 1 + N_{10}\)
- Given \(X_0 = 1\)

\[ N_{10} = \begin{cases} 1 & \text{if } X_1 = 0 \\ 1 + N_{21} + N_{10}' & \text{if } X_1 = 2 \end{cases} \]  \hspace{1cm} (3)

Note \(N_{21}\) and \(N_{10}'\) are independent and have the same distribution as \(N_{10}\) (Why?)

Generating Function of \(N_{10}\)

Let \(G(s)\) be the generating function of \(N_{10}\). From (??), we know that

\[ G(s) = \frac{1}{2}s + \frac{1}{2}E[s^{N_{10}}] = \frac{1}{2}s + \frac{1}{2}sG(s)^2 \]

which is a quadratic equation in \(G(s)\). The two roots are

\[ G(s) = \frac{1 \pm \sqrt{1 - s^2}}{s} \]

Since \(G(s)\) must lie between 0 and 1 when \(0 < s < 1\). So

\[ G(s) = \frac{1 - \sqrt{1 - s^2}}{s} \]

Note that

\[ G'(s) = \frac{1}{\sqrt{1 - s^2} + 1 - s^2} \quad \text{E}[N_{10}] = \lim_{s \to 1^-} G'(s) = \infty \]

which implies \(\text{E}[N_{00}] = 1 + \text{E}[N_{10}] = \infty\).

\(\Rightarrow\) Symmetric random walk is null recurrent.

The power series expansion of \(G(s) = \frac{1 - \sqrt{1 - s^2}}{s}\) can be found via Newton’s binomial formula

\[(1 - s^2)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-s^2)^k\]

where \(\binom{\alpha}{0} = 1\) and for \(k \geq 1\), \(\binom{\alpha}{k} = \prod_{i=0}^{k-1} (\alpha - i)/k!\).

\[ \binom{1/2}{k} = \frac{1}{2 \cdot \Gamma(1/2)} \prod_{i=0}^{k-1} (\frac{3}{2} - i) \]

\[ = \frac{(-1)^{k-1} \cdot 1 \cdot 3 \cdot \ldots \cdot (2k - 3) \cdot (2k - 1)}{2^k k!} \]

\[ = \frac{(-1)^{k-1} \cdot 1 \cdot 3 \cdot \ldots \cdot (2k - 3) (2k - 1)}{2^k k! (2k - 1)} \]

\[ = \frac{(-1)^{k-1} \frac{(2k-1)!}{2^{k-1}(k-1)!}}{2^k k! (2k - 1)} = \frac{(-1)^{k-1} (2k - 1)}{2^{2k-1}(2k - 1)} \binom{2k - 1}{k} \]

From all the above we have

\[ G(s) = \frac{1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-s^2)^k}{s} = -\sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} s^{2k-1} \]

\[ = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}(2k - 1)} \binom{2k-1}{k} s^{2k-1} \]

So the distribution of \(N_{10}\) is

\[ P(N_{10} = 2k - 1) = \frac{1}{2^{2k-1}(2k - 1)} \binom{2k-1}{k} \]

\[ P(N_{10} = 2k) = 0 \]

for \(k = 0, 1, 2, \ldots\).
4.7 Branching Processes

Recall a Branching Process is a population of individuals in which

- all individuals have the same lifespan, and
- each individual will produce a random number of offsprings at the end of its life

Let \( X_n \) = size of the \( n \)th generation, \( n = 0, 1, 2, \ldots \). Let \( Z_{n,i} \) = # of offsprings produced by the \( i \)th individuals in the \( n \)th generation. Then

\[
X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i} \quad (4)
\]

Suppose \( Z_{n,i} \)'s are i.i.d with probability mass function

\[
P(Z_{n,i} = j) = P_j, \ j \geq 0.
\]

We suppose the non-trivial case that \( P_j < 1 \) for all \( j \geq 0 \).

\( \{X_n\} \) is a Markov chain with state space = \{0, 1, 2, \ldots \}.

Variance of a Branching Process

Let \( \sigma^2 = \text{Var}(Z_{n,i}) \). Again from the fact that \( X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \), we have

\[
\text{E}[X_n|X_{n-1}] = X_{n-1} \mu, \ \ \text{Var}(X_n|X_{n-1}) = X_{n-1} \sigma^2
\]

and hence

\[
\text{Var}(\text{E}[X_n|X_{n-1}]) = \text{Var}(X_{n-1} \mu) = \mu^2 \text{Var}(X_{n-1})
\]

\[
\text{E}[\text{Var}(X_n|X_{n-1})] = \sigma^2 \text{E}[X_{n-1}] = \sigma^2 \mu^{n-1}.
\]

So

\[
\begin{align*}
\text{Var}(X_n) &= \text{E}[\text{Var}(X_n|X_{n-1})] + \text{Var}(\text{E}[X_n|X_{n-1}]) \\
&= \sigma^2 \mu^n + \mu^2 \text{Var}(X_{n-1}) \\
&= \sigma^2 \mu^n + \mu^2 \text{Var}(X_{n-1}) \\
&= \sigma^2 \mu^{n-1} + \mu^n + \ldots + \mu^{2n-2} + \mu^{2n} \text{Var}(X_0) \\
&= \mu^{2n} \text{Var}(X_0) + \left\{ \begin{array}{ll}
\sigma^2 \mu^{n-1} \left( \frac{1-\mu^n}{1-\mu} \right) & \text{if } \mu \neq 1 \\
\frac{n \sigma^2}{2} & \text{if } \mu = 1
\end{array} \right.
\end{align*}
\]

Mean of a Branching Process

Let \( \mu = \text{E}[Z_{n,i}] = \sum_{j=0}^{\infty} j P_j \). Since \( X_n = \sum_{i=1}^{X_{n-1}} Z_{n-1,i} \), we have

\[
\text{E}[X_n|X_{n-1}] = \text{E} \left[ \sum_{i=1}^{X_{n-1}} Z_{n-1,i} \right] = X_{n-1} \text{E}[Z_{n-1,i}] = X_{n-1} \mu
\]

So

\[
\text{E}[X_n] = \text{E}[\text{E}[X_n|X_{n-1}]] = \text{E}[X_{n-1}\mu] = \mu \text{E}[X_{n-1}]
\]

If \( X_0 = 1 \), then

\[
\text{E}[X_n] = \mu^2 \text{E}[X_{n-2}] = \ldots = \mu^n \text{E}[X_0] = \mu^n
\]

- If \( \mu < 1 \Rightarrow \text{E}[X_n] \rightarrow 0 \) as \( n \rightarrow \infty \Rightarrow \lim_{n \rightarrow \infty} P(X_n \geq 1) = 0 \) the branching processes will eventually die out.

- What if \( \mu = 1 \) or \( \mu > 1 \)?

Generating Functions of the Branching Processes

Let \( g(s) = \text{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k \) be the generating function of \( Z_{n,i} \), and \( G_n(s) \) be the generating function of \( X_n, n = 0, 1, 2, \ldots \).

Then \( \{G_n(s)\} \) satisfies the following two iterative equation

(i) \( G_{n+1}(s) = G_n(g(s)) \) for \( n = 0, 1, 2, \ldots \)

(ii) \( G_{n+1}(s) = g(G_n(s)) \) if \( X_0 = 1 \), for \( n = 0, 1, 2, \ldots \)

Proof of (i).

\[
\begin{align*}
\text{E}[s^{X_{n+1}}|X_n] &= \text{E} \left[ s^{\sum_{i=1}^{X_n} Z_{n,i}} \right] \text{E} \left[ \prod_{i=1}^{X_n} s^{Z_{n,i}} \right] \\
&= \prod_{i=1}^{X_n} \text{E}[s^{Z_{n,i}}] \text{ (by indep. of } Z_{n,i}) \\
&= \prod_{i=1}^{X_n} g(s) = g(s)^{X_n}
\end{align*}
\]

From which, we have

\[
G_{n+1}(s) = \text{E}[s^{X_{n+1}}|X_n] = \text{E} \left[ \text{E}[s^{X_{n+1}|X_n}] \right] = \text{E}[g(s)^{X_n}] = G_n(g(s))
\]

since \( G_n(s) = \text{E}[s^{X_n}] \).

Proof of (ii): HW today
Proposition I

Let \( \mu = \mathbb{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_{j} \). If \( \mu \leq 1 \), the extinction probability \( \pi_{0} \) is 1 unless \( P_{1} = 1 \).

Proof. Let \( h(s) = g(s) - s \). Since \( g(1) = 1 \), \( g'(1) = \mu \),

\[
h(1) = g(1) - 1 = 0, \]
\[
h'(s) = \left( \sum_{j=1}^{\infty} jP_{j}s^{j-1} \right) - 1 \leq \left( \sum_{j=1}^{\infty} jP_{j} \right) - 1 = \mu - 1 \quad \text{for } 0 \leq s < 1
\]

Thus \( \mu \leq 1 \Rightarrow h'(s) \leq 0 \) for \( 0 \leq s < 1 \)

\[
\Rightarrow h(s) \text{ is non-increasing in } [0,1]
\]
\[
\Rightarrow h(s) > h(1) = 0 \text{ for } 0 \leq s < 1
\]
\[
\Rightarrow g(s) > s \quad \text{for } 0 \leq s < 1
\]
\[
\Rightarrow \text{There is no root in } [0,1).
\]

Proposition II

If \( \mu > 1 \), there is a unique root of the equation \( g(s) = s \) in the domain \( [0,1) \), and that is the extinction probability.

Proof. Let \( h(s) = g(s) - s \). Observe that

\[
h(0) = g(0) = P_{0} > 0
\]
\[
h'(0) = g'(0) - 1 = P_{1} - 1 < 0
\]

Then \( \mu > 1 \Rightarrow h'(1) = \mu - 1 > 0 \)

\[
\Rightarrow h(s) \text{ is increasing near } 1
\]
\[
\Rightarrow h(1 - \delta) < h(1) = 0 \text{ for } \delta > 0 \text{ small enough}
\]

Since \( h(s) \) is continuous in \( [0,1) \), there must be a root to

\[
h(s) = s. \quad \text{The root is unique since}
\]
\[
h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j-1)P_{j}s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1
\]

\( h(s) \) is convex in \( [0,1) \).