STAT 25100 Lecture 8
4.3 Expected Value of Discrete Random Variables

Yibi Huang
Department of Statistics
University of Chicago

The Expected Value of a Random Variable

Definition: Expected Value of a Discrete Random Variable

Let X be a discrete random variable with PMF $p(x)$.
The **expected value** or the **expectation** or the **mean** of X, denoted by $\mathrm{E}[X]$, or μ_x is a weighted average of the possible values of X, where the weights are the probabilities of those values.

$$\mu_x = \mathrm{E}[X] = \sum_{\text{all values of } x} x \cdot p(x)$$

provided that provided that $\sum_x |x| p(x) < \infty$.

If the sum diverges, the expectation is undefined.

Expected Value of the Geometric Distribution

Recall the Geometric PMF is

$$p(k) = (1-p)^{x-1}p \quad \text{for } x = 1, 2, 3, \ldots .$$

To evaluate its expected value

$$\mathrm{E}(X) = \sum\nolimits_x x \cdot p(x) = \sum\nolimits_{x=1}^{\infty} x(1-p)^{x-1}p,$$

we'll start from the geometric series

$$\sum_{x=0}^{\infty} r^x = \frac{1}{1-r} \text{ if } |r| < 1 \Rightarrow \begin{array}{c}\text{differentiate}\\ \text{both sides}\\ \text{w/ respect to } r\end{array} \Rightarrow \sum_{x=1}^{\infty} x r^{x-1} = \frac{1}{(1-r)^2}, \text{ for } |r| < 1.$$

Applying the new identity with $r = 1 - p$, we get

$$\mathrm{E}(X) = \underbrace{\sum\nolimits_{x=1}^{\infty} x(1-p)^{x-1}}_{=1/(1-(1-p))^2} p = \frac{1}{p^2} \cdot p = \frac{1}{p}.$$

Expected Value of Negative Binomial

Recall the Negative Binomial PMF is $p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad x = r, r+1, \ldots$.

The expected value of the Negative Binomial is

$$\mathrm{E}(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= \sum_{x=r}^{\infty} r \binom{x}{r} p^r (1-p)^{x-r} \ \Big(\text{since } x\binom{x-1}{r-1} = \frac{x \cdot (x-1)!}{(r-1)!(x-r)!} = \frac{x!}{(r-1)!(x-r)!} = r\Big($$

$$= \frac{r}{p} \sum_{x=r}^{\infty} \binom{x}{r} p^{r+1} (1-p)^{x-r}$$

$$= \frac{r}{p} \underbrace{\sum_{y=r+1}^{\infty} \binom{y-1}{r+1-1} p^{r+1} (1-p)^{y-(r+1)}}_{=1, \text{ since it's the sum of PMF for NB}(r+1,p)} \quad (\text{let } y = x+1)$$

$= \frac{r}{\ }$ **Intuition:** (As it takes $1/p$ trials on average to get the first success,

An Example Where $\mathrm{E}(X)$ Is Infinite

Game: Toss a fair coin repeatedly, the longer one keeps getting heads, the more reward. Specifically, the reward is 2^n cents if one gets n consecutive Heads before the first Tail. The PMF of the reward X (in cents) is

$$\mathrm{P}(X = 2^n) = \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \dots .$$

The expected value of X is

$$\mathrm{E}(X) = \sum_{n=0}^{\infty} 2^n \mathrm{P}(X = 2^n) = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} = \infty.$$

Expected Values of Functions of Random Variables

Example: Expected Values of X^2 (1)

If X is a random variable with the PMF $p_X(x)$ below, how to find the expected value of $Y = X^2$?

x	-1	0	1	2
$p_X(x) = \mathrm{P}(X = x)$	0.3	0.4	0.2	0.1

Method 1: First we find the PMF $p_Y(y)$ of Y.

- Possible values of Y:
- $\mathrm{P}(Y = 0) = \mathrm{P}(X^2 = 0) = \mathrm{P}(X = 0) = 0.4$
- $\mathrm{P}(Y = 1) = \mathrm{P}(X^2 = 1) =$
- $\mathrm{P}(Y = 4) = \mathrm{P}(X^2 = 4) = \mathrm{P}(X = 2) = 0.1$
- PMF of Y:

y	0	1	4
$p_Y(y)$	0.4	0.5	0.1

- Expected value of $Y = X^2$ is thus

$$\mathrm{E}(Y) = \sum_y y p_Y(y) = 0 \cdot 0.4 + 1 \cdot 0.5 + 4 \cdot 0.1 = 0.9$$

Example: Expected Values of X^2 (2)

x	-1	0	1	2
$p_X(x) = \mathrm{P}(X = x)$	0.3	0.4	0.2	0.1

Method 2: In fact, we can find expected value of $Y = X^2$ using the PMF of X directly as

$$\begin{aligned} \mathrm{E}(Y) = \mathrm{E}(X^2) &= \sum_x x^2 \cdot p_X(x) \\ &= (-1)^2 \cdot 0.3 + 0^2 \cdot 0.4 + 1^2 \cdot 0.2 + 2^2 \cdot 0.1 \\ &= 0.9 \end{aligned}$$

- Same answer as Method 1
- Method 2 is easier no need to find the PMF of $Y = X^2$,

Expected Values of Functions of Random Variables

Generally, if X is a random variable with PMF $p_X(x)$, and $Y = g(X)$, how to find the expected value of Y?

Expected Values of Functions of Random Variables

Generally, if X is a random variable with PMF $p_X(x)$, and $Y = g(X)$, how to find the expected value of Y?

Method 1: Find the PMF $p_Y(y)$ of Y and then calculate the expected value as

$$\mathrm{E}(Y) = \sum_y y p_Y(y)$$

Method 2: One can calculate $\mathrm{E}(Y)$ directly using the PMF X as

$$\mathrm{E}(Y) = \mathrm{E}(g(X)) = \sum_x g(x) p_X(x)$$

- Method 2 is easier since one doesn't have to find the PMF of $Y = g(X)$, which is sometimes not easy

Proof of the Equivalence of the Two Methods

First note that

$$p_Y(y) = \mathrm{P}(Y = y) = \mathrm{P}\Big(\bigcup_{x:\ g(x)=y} \{X = x\}\Big) = \sum_{x:\ g(x)=y} \mathrm{P}(X = x) = \sum_{x:\ g(x)=y} p_X(x)$$

Thus

$$\begin{aligned}
\mathrm{E}(Y) = \sum_y y p_Y(y) &= \sum_y y \sum_{x:\ g(x)=y} p_X(x) \\
&= \sum_y \sum_{x:\ g(x)=y} y p_X(x) \\
&= \sum_y \sum_{x:\ g(x)=y} g(x) p_X(x) \\
&= \sum_x g(x) p_X(x)
\end{aligned}$$

Expected Value of $aX + b$

If X is a random variable, the expected value for its Linear transformation $Y = g(X) = aX + b$ is

$$\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b.$$

Proof.

$$\begin{aligned}
\mathrm{E}(aX + b) &= \sum_x (ax + b)p(x) \\
&= \sum_x (ax\,p(x) + bp(x)) \\
&= \sum_x axp(x) + \sum_x bp(x) \\
&= a \underbrace{\sum_x xp(x)}_{=\mathrm{E}(X)} + b \underbrace{\sum_x p(x)}_{=1} \\
&= a\,\mathrm{E}(X) + b
\end{aligned}$$

$\mathrm{E}[h_1(X) + h_2(X)] = \mathrm{E}[h_1(X)] + \mathrm{E}[h_2(X)]$

If X is a random variable, and $g(X)$ is a function of X. Suppose $g(x)$ is the sum of two functions

$$g(x) = h_1(x) + h_2(x)$$

then

$$\mathrm{E}(g(x)) = \mathrm{E}[h_1(X) + h_2(X)] = \mathrm{E}[h_1(X)] + \mathrm{E}[h_2(X)]$$

Proof.

$$\begin{aligned}
\mathrm{E}(h_1(X) + h_2(X)) &= \sum_x (h_1(x) + h_2(x))p(x) \\
&= \sum_x h_1(x)p(x) + h_2(x)p(x) \\
&= \sum_x h_1(x)p(x) + \sum_x h_2(x)p(x) \\
&= \mathrm{E}(h_1(X)) + \mathrm{E}(h_2(X))
\end{aligned}$$

Variance, Standard Deviation, and Various Moments

Variance & Standard Deviation (SD)

One measure of spread of a random variable (or its probability distribution) is the *variance*.

The **variance** of a random variable X, denoted as $\mathrm{Var}(X)$ or σ_X^2 is defined as the average squared distance from the expected value $\mu_X = \mathrm{E}(x)$.

$$\begin{aligned} \mathrm{Var}(X) &= \sigma^2 = \text{"sigma squared"} \\ &= \mathrm{E}\left[(X - \mu_X)^2\right] \\ &= \sum_x (x - \mu_X)^2 p_X(x) \end{aligned}$$

provided that the variance is $< \infty$.

Square root of the variance is the *standard deviation (SD).*

$$\mathrm{SD}(X) = \sigma = \sqrt{\mathrm{Var}(X)}$$

Variance of $aX + b$

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Variance of $aX + b$

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Taking expected value of the above we get

$$\begin{array}{ccc} \mathrm{E}[Y - \mathrm{E}(Y)]^2 & = & \mathrm{E}[a^2(X - \mu)^2] \\ \| & & \| \\ \mathrm{Var}(Y) & & a^2\,\mathrm{E}[(X - \mu)^2] \\ \| & & \| \\ \mathrm{Var}(aX + b) & & a^2\,\mathrm{Var}(X) \end{array}$$

This shows that

$$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X).$$

Moments & Central Moments

Given a random variable X with mean μ

- its *kth moment* is defined to be $\mathrm{E}[X^k]$, and
- its *kth centeral moment* is defined to be $\mathrm{E}[(X-\mu)^k]$,

provided that $\mathrm{E}[|X|^k] < \infty$ and $\mathrm{E}[|X-\mu|^k] < \infty$.

Note that

- the 1st moment $\mathrm{E}(X)$ is the mean $=$ expected value
- the 1st central moment $\mathrm{E}(X-\mu)$ is always 0
- the 2nd central moment $\mathrm{E}[(X-\mu)^2]$ is the variance.

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum_x (x-\mu)^2\, p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum_x x\, p(x)}_{=\mu} + \mu^2 \underbrace{\sum_x p(x)}_{=1} \\
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 = \mathrm{E}(X^2) - \mu^2
\end{aligned}$$

Example of Variance

Find the variance for a random variable X with the PMF below.

x	-1	0	1	2
$p_X(x)$	0.3	0.4	0.2	0.1

Sol #1: Use the definition of variance.

$$\mu = \mathrm{E}(X) = \sum_x x p_X(x) = (-1)\cdot 0.3 + 0 \cdot 0.4 + 1 \cdot 0.2 + 2 \cdot 0.1 = 0.1$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X-\mu)^2 = \sum_x (x-0.1)^2 p_X(x) \\ &= (-1-0.1)^2 \cdot 0.3 + (0-0.1)^2 \cdot 0.4 + (1-0.1)^2 \cdot 0.2 + (2-0.1)^2 \cdot 0.1 \\ &= 1.21 \cdot 0.3 + 0.01 \cdot 0.4 + 0.81 \cdot 0.2 + 3.61 \cdot 0.1 \\ &= 0.89 \end{aligned}$$

x	-1	0	1	2
$p_X(x)$	0.3	0.4	0.2	0.1

Sol #2: Use the shortcut formula.

$$\begin{aligned}
\mathrm{E}(X) &= \sum_x x p_X(x) = (-1)\cdot 0.3 + 0\cdot 0.4 + 1\cdot 0.2 + 2\cdot 0.1 = 0.1 \\
\mathrm{E}(X^2) &= \sum_x x^2 \cdot p_X(x) \\
&= (-1)^2\cdot 0.3 + 0^2\cdot 0.4 + 1^2\cdot 0.2 + 2^2\cdot 0.1 = 0.9 \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 = 0.9 - 0.1^2 = 0.89
\end{aligned}$$

- same answer & easier to calculate!

Factorial Moments

Given a random variable X with mean μ, its *kth moment* is defined to be

$$E[X(X-1)\cdots(X-k+1)], \quad k=1,2,3,\ldots$$

- first factorial moment $= E(X) =$ first moment
- For Binomial, Hypergeometric, and Poisson, Factorial Moments are easier to calculate than Moments

Factorial Moments of Poisson

Recall the Poisson PMF is $p(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$.

$$
\begin{aligned}
&\mathrm{E}[X(X-1)\cdots(X-k+1)] \\
&= \sum_{x=0}^{\infty} x(x-1)\cdots(x-k+1)e^{-\lambda}\frac{\lambda^x}{x!} \\
&= \sum_{x=k}^{\infty} x(x-1)\cdots(x-k+1)e^{-\lambda}\frac{\lambda^x}{x!} \qquad \text{first } k \text{ terms are } 0 \\
&= \sum_{x=k}^{\infty} e^{-\lambda}\frac{\lambda^x}{(x-k)!} \\
&= \lambda^k e^{-\lambda}\sum_{x=k}^{\infty}\frac{\lambda^{x-k}}{(x-k)!} \\
&= \lambda^k e^{-\lambda}\underbrace{\sum_{y=0}^{\infty}\frac{\lambda^y}{y!}}_{=e^{\lambda}} \qquad \text{let } y = x-k \\
&= \lambda^k
\end{aligned}
$$

Mean & Variance of Poisson

For $X \sim \text{Poisson}(\lambda)$, the kth factorial moments are

$$\mathrm{E}[X(X-1)\cdots(X-k+1)] = \lambda^k, \quad k = 1, 2, 3, \ldots$$

- $k = 1$: $\mathrm{E}(X) = \lambda$
- $k = 2$: $\mathrm{E}(X(X-1)) = \lambda^2$
- To find the 2nd moment $\mathrm{E}(X^2)$, observe that $X^2 = X(X-1) + X$. Thus

$$\mathrm{E}(X^2) = \mathrm{E}(X(X-1)) + \mathrm{E}(X) = \lambda^2 + \lambda$$

- $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$

Factorial Moments of Binomial (2)

Recall the Binomial PMF is $p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \le x \le n.$

$$\begin{aligned}
&\mathrm{E}[X(X-1)\cdots(X-k+1)] \\
&= \sum_{x=0}^{n} x(x-1)\cdots(x-k+1)\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=k}^{n} x(x-1)\cdots(x-k+1)\binom{n}{x} p^x (1-p)^{n-x} \quad \text{first } k \text{ terms are } 0
\end{aligned}$$

Key step:

$$\begin{aligned}
x(x-1)\cdots(x-k+1)\binom{n}{x} &= x(x-1)\cdots(x-k+1)\frac{n!}{x!(n-x)!} \\
&= \frac{n!}{(x-k)!(n-x)!} \\
&= n(n-1)\cdots(n-k+1)\binom{n-k}{x-k}.
\end{aligned}$$

Factorial Moments of Binomial (2)

$$\mathrm{E}[X(X-1)\cdots(X-k+1)]$$

$$\sum_{x=k}^{n} x(x-1)\cdots(x-k+1)\binom{n}{x}p^x(1-p)^{n-x}$$

$$=\sum_{x=k}^{n} n(n-1)\ldots(n-k+1)\binom{n-k}{x-k}p^x(1-p)^{n-x} \quad \text{(key step in previous page)}$$

$$=n(n-1)\ldots(n-k+1)p^k\sum_{x=k}^{n}\binom{n-k}{x-k}p^{x-k}(1-p)^{n-x}$$

$$=n(n-1)\ldots(n-k+1)p^k\underbrace{\sum_{y=0}^{n-k}\binom{n-k}{y}p^y(1-p)^{n-k-y}}_{=(p+1-p)^{n-k}=1} \quad \text{let } y=x-k$$

$$=n(n-1)\ldots(n-k+1)p^k \quad \left(\text{Binomial expansion} \quad (a+b)^N=\sum^{N}\binom{N}{i}a^ib^{N-i} \text{ with } \begin{array}{l} a=p \\ b=1-p \end{array}\right.$$

Mean & Variance of Binomial

For $X \sim \text{Bin}(n, p)$, the kth factorial moments are

$$\mathrm{E}[X(X-1)\cdots(X-k+1)] = n(n-1)\ldots(n-k+1)p^k, \quad k = 1, 2, 3, \ldots$$

- $k = 1$: $\mathrm{E}(X) = np$
- $k = 2$: $\mathrm{E}(X(X-1)) = n(n-1)p^2$
- To find the 2nd moment $\mathrm{E}(X^2)$, observe that $X^2 = X(X-1) + X$. Thus

$$\mathrm{E}(X^2) = \mathrm{E}(X(X-1)) + \mathrm{E}(X) = n(n-1)p^2 + np$$

- $$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p).$$

Factorial Moments of Hypergeometric

Hypergeometric PMF:

$$p(x) = \binom{R}{x}\binom{B}{d-x}\Big/\binom{R+B}{d}, \quad 0 \le x \le R,\ 0 \le d-x \le B.$$

Factorial Moment:

$$\mathrm{E}[X(X-1)\cdots(X-k+1)] = \sum_{x=0}^{R} x(x-1)\cdots(x-k+1)\binom{R}{x}\binom{B}{d-x}\Big/\binom{R+B}{d}.$$

We first calculate its numerator on the next page

$$\sum_{x=0}^{R} x(x-1)\cdots(x-k+1)\binom{R}{x}\binom{B}{d-x}$$

$$= \sum_{x=k}^{R} x(x-1)\cdots(x-k+1)\binom{R}{x}\binom{B}{d-x}$$

$$= \sum_{x=k}^{R} R(R-1)\cdots(R-k+1)\binom{R-k}{x-k}\binom{B}{d-x} \quad \text{(same key step as Binomial)}$$

$$= R(R-1)\cdots(R-k+1)\sum_{i=0}^{R-k}\binom{R-k}{i}\binom{B}{d-k-i} \quad (\text{let } i = x-k)$$

$$= R(R-1)\cdots(R-k+1)\binom{R+B-k}{d-k} \quad \text{(Vandermonde's identity)}$$

Putting back the denominator $\binom{R+B}{d}$, we get the factorial moment

$$\mathrm{E}[X(X-1)\cdots(X-k+1)] = R(R-1)\cdots(R-k+1)\binom{R+B-k}{d-k} \Big/ \binom{R+B}{d}$$

Mean & Variance of Hypergeometric

For $X \sim \text{Hypergeometric}(R, B, d)$, the kth factorial moments, $k = 1, 2, 3, \dots,$ are

$$\mathrm{E}[X(X-1)\cdots(X-k+1)] = R(R-1)\cdots(R-k+1)\binom{R+B-k}{d-k}\Big/\binom{R+B}{d}$$
$$= \frac{R(R-1)\cdots(R-k+1)\times d(d-1)\cdots(d-k+1)}{(R+B)(R+B-1)\cdots(R+B-k+1)}$$

- $k=1$: $\mathrm{E}(X) = \dfrac{dR}{R+B}$
- $k=2$: $\mathrm{E}(X(X-1)) = \dfrac{R(R-1)d(d-1)}{(R+B)(R+B-1)}$
- To find the 2nd moment $\mathrm{E}(X^2)$, observe that $X^2 = X(X-1) + X$. Thus

$$\mathrm{E}(X^2) = \mathrm{E}(X(X-1)) + \mathrm{E}(X) = \frac{R(R-1)d(d-1)}{(R+B)(R+B-1)} + \frac{dR}{R+B}$$
$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$
$$= \frac{R(R-1)d(d-1)}{(R+B)(R+B-1)} + \frac{Rd}{(R+B)} - \frac{R^2d^2}{(R+B)^2} = d\frac{RB(R+B-d)}{(R+B)^2(R+B-1)}.$$

Variance for Negative Binomial (1)

Negative Binomial PMF: $p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad x = r, r+1, \ldots.$

For Negative Binomial, $\mathrm{E}(X(X+1))$ is easier to calculate than $\mathrm{E}(X^2)$ or $\mathrm{E}(X(X-1))$.

$$\mathrm{E}(X(X+1)) = \sum_{x=r}^{\infty} x(x+1) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Key step:

$$x(x+1)\binom{x-1}{r-1} = \frac{(x+1)x \cdot (x-1)!}{(x-r)!(r-1)!} = \frac{(x+1)!}{(x-r)!{\color{red}(r-1)!}} = \frac{{\color{red}r(r+1)} \cdot (x+1)!}{(x-r)!{\color{red}(r+1)!}}$$

$$= r(r+1)\binom{x+1}{r+1}.$$

Variance for Negative Binomial (2)

$$
\begin{aligned}
\mathrm{E}(X(X+1)) &= \sum_{x=r}^{\infty} x(x+1)\binom{x-1}{r+1} p^r (1-p)^{x-r} \\
&= \sum_{x=r}^{\infty} r(r+1)\binom{x+1}{r+1} p^r (1-p)^{x-r} \quad \text{(key step on previous page)} \\
&= \frac{r(r+1)}{p^2} \sum_{x=r}^{\infty} \binom{x+1}{r+1} p^{r+2} (1-p)^{x-r} \\
&= \frac{r(r+1)}{p^2} \underbrace{\sum_{y=r+2}^{\infty} \binom{y-1}{r+2-1} p^{r+2} (1-p)^{y-(r+2)}}_{=1, \text{ since it's the sum of PMF for NB}(r+2,p)} \quad (\text{let } y = x+2) \\
&= \frac{r(r+1)}{p^2}.
\end{aligned}
$$

Variance for Negative Binomial (3)

The second moment:

$$\mathrm{E}(X^2) = \mathrm{E}(X(X+1)) - \mathrm{E}(X) = \frac{r(r+1)}{p^2} - \frac{r}{p}.$$

By the shortcut formula, the variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$

Summary: Common Discrete Distributions — Definitions

Distribution	Genesis
Bernoulli (p)	1 for Success, 0 for Failure, for the outcome of one Bernoulli trial
Binomial(n, p)	number of successes obtained in n Bernoulli trials, that each trial has a success probability p
Geometric(p)	number of Bernoulli trials needed to get the first success, that each trial has a success probability p
Negative Binomial(r, p)	number of Bernoulli trials needed to get r successes, that each trial has a success probability p
Hypergeometric (R, B, d)	number of red marbles obtained in d draws without replacement from a box with R red and B blue marbles.
Poisson(λ)	limit of Binomial(n, p) as $n \to \infty$, $p \to 0$, and $np \to \lambda$

Summary: Common Discrete Distributions — PMF, Mean, Variance

Name and range	PMF at x	Mean	Variance
Bernoulli(p) on $\{0,1\}$	$\begin{cases} 1-p & \text{if } x=0 \\ p & \text{if } x=1 \end{cases}$	p	$p(1-p)$
Binomial(n,p) on $\{0,1,\ldots,n\}$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Geometric(p) on $\{1,2,3\ldots\}$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial(r,p) on $\{r, r+1, r+2, \ldots\}$	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Hypergeometric(R,B,d) on $\{0,1,2,\ldots,\min(d,R)\}$	$\binom{R}{x}\binom{B}{d-x} \Big/ \binom{R+B}{d}$	$d\frac{R}{R+B}$	$d\frac{RB(R+B-d)}{(R+B)^2(R+B-1)}$
Poisson(λ) on $\{0,1,2,\ldots\}$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ

Summary: Relations Between Discrete Distributions

- Bernoulli & Binomial
  - Binomial$(n = 1, p)$ = Bernoulli(p)
  - If $X_1, \ldots, X_n$ are i.i.d. Bernoulli(p), then $\sum_{i=1}^{n} X_i \sim$ Binomial(n, p)
  - If $X \sim$ Binomial(m, p) and $Y \sim$ Binomial(n, p) are independent, then $X + Y \sim$ Binomial$(m + n, p)$
- Geometric & Negative Binomial
  - Negative Binomial$(r = 1, p)$ = Geometric(p)
  - If $X_1, \ldots, X_n$ are i.i.d. Geometric(p), then $\sum_{i=1}^{n} X_i \sim$ Negative Binomial$(r = n, p)$
  - If $X \sim$ NegBin(m, p) and $Y \sim$ NegBin(n, p) are independent, then $X + Y \sim$ NegBin$(m + n, p)$
- If $R \gg d$ and $B \gg d$,

$$\text{Hypergeometric}(R, B, d) \approx \text{Binomial}\left(n = d, p = \frac{R}{R + B}\right)$$

- If $n \to \infty$, $p \to 0$, and $np \to \lambda$, Binomial$(n, p) \approx$ Poisson$(\lambda = np)$
- If $X_i \sim$ Poisson(λ_i) are independent, then $\sum_{i=1}^{n} X_i \sim$ Poisson$(\sum_{i=1}^{n} \lambda_i)$