STAT 24400 Lecture 11
Section 4.5 Moment Generating Functions (MGF)

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Moment Generating Function (MGF)

The *moment generating function (MGF)* $M(t)$ of the random variable X is defined to be

$$M(t) = \mathrm{E}(e^{tX}) = \begin{cases} \sum_x e^{tx} p_X(x) & \text{if discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) \mathrm{d}x & \text{if continuous} \end{cases}$$

where $p_X(x)$ and $f_X(x)$ are the PMF/PDF of X.

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where $p_X(x)$ and $f_X(x)$ are the PMF/PDF of X.

- MGF $M(t)$ is NOT a single value but a function of t
- $M(0) = \mathrm{E}(e^{0\dot{X}}) = \mathrm{E}[1] = 1$ always

Example: MGF for Geometric

For **Geometric**(p), the PMF is

$$p(x) = (1-p)^x p, \quad x = 0, 1, 2, \dots,$$

Its MGF is

$$\begin{aligned} M(t) &= \sum_{x=0}^{\infty} e^{tx} p(x) \\ &= \sum_{x=0}^{\infty} e^{tx} (1-p)^x p \\ &= p \sum_{x=0}^{\infty} (e^t (1-p))^x \\ &= \frac{p}{1-(1-p)e^t}, \quad \text{since } \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}. \end{aligned}$$

The last step is valid only when $(1-p)e^t < 1$, or $t < -\log(1-p)$. Thus the MGF is defined when $(1-p)e^t < 1$

Example: MGF for Binomial

For **Binomial**(n, p), the PMF is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad k = 0, 1, 2, \ldots, n.$$

Its MGF is

$$\begin{aligned} M(t) = \sum_{x=0}^{n} e^{tx} p(x) &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= (pe^t + (1-p))^n \quad \text{valid for } -\infty < t < \infty. \end{aligned}$$

The last step comes from the Binomial expansion $(a+b)^N = \sum_{x=0}^{N} \binom{N}{x} a^x b^{N-x}$ for $a = pe^t$, $b = 1-p$, and $N = n$.

Example: MGF for Exponential

For Exponential(λ), the PDF is

$$f(x) = \lambda e^{-\lambda x}, \quad 0 \leq x < \infty.$$

Its MGF is

$$\begin{aligned} M(t) = \int_0^\infty e^{tx} f(x) \mathrm{d}x &= \int_0^\infty e^{tx} \lambda e^{-\lambda x} \mathrm{d}x \\ &= \int_0^\infty \lambda e^{-(\lambda - t)x} \mathrm{d}x \\ &= \frac{\lambda}{\lambda - t}, \end{aligned}$$

The integral is finite only when $\lambda - t > 0$.
Thus the MGF is defined only when $-\infty < t < \lambda$.

MGF for Standard Normal $N(0,1)$

The PDF for $N(0,1)$ is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for $-\infty \le x < \infty$.
Its MGF is thus

$$M(t) = \int_{-\infty}^{\infty} e^{tx}\phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\overbrace{-x^2/2+tx}^{\text{see below}}}\mathrm{d}x$$

Using the technique of completing the square, as

$$-\frac{x^2}{2} + tx = -\frac{1}{2}(x^2 - 2tx + t^2) + \frac{t^2}{2} = -\frac{1}{2}(x-t)^2 + \frac{t^2}{2},$$

the integral equals

$$M(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2+t^2/2}\mathrm{d}x = e^{t^2/2}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\mathrm{d}x}_{=1} = e^{t^2/2}.$$

The last integral is 1 since it integrates over PDF of $N(t,1)$.
The MGF is defined for all $-\infty < t < \infty$.

Moments and Moment Generating Functions

As $M(t) = \mathrm{E}(e^{tX})$, its first derivative is

$$M'(s) = \frac{d}{\mathrm{d}t}\,\mathrm{E}[e^{tX}] = \mathrm{E}\left[\frac{d}{\mathrm{d}t}e^{tX}\right] = \mathrm{E}[Xe^{tX}].$$

Its second derivative is

$$M''(s) = \frac{d}{\mathrm{d}t}M'(t) = \frac{d}{\mathrm{d}t}\,\mathrm{E}[Xe^{tX}] = \mathrm{E}\left[\frac{d}{\mathrm{d}t}Xe^{tX}\right] = \mathrm{E}[X^2 e^{tX}].$$

In general, the kth derivative of the MGF is

$$M^{(k)}(s) = \frac{d^k}{\mathrm{d}t^k}M(t) = \mathrm{E}[X^k e^{tX}].$$

Plugging in $t = 0$, we get

$$M'(0) = \mathrm{E}(X), \quad M''(0) = \mathrm{E}(X^2), \quad M^{(k)}(0) = \mathrm{E}(X^k), \ldots,$$

Moment generating functions got the name since the moments of X can be obtained by successively differentiating $M(t)$.

Example: Calculating Moments Using MGF — Exponential

The MGF for Exponential(p) is

$$M(t) = \frac{\lambda}{\lambda - t}.$$

The derivatives and the moments are thus

$$\begin{aligned}
M'(t) &= \frac{\lambda}{(\lambda - t)^2} \quad &\Rightarrow \quad \mathrm{E}(X) = M'(0) = \frac{1}{\lambda} \\
M''(t) &= \frac{2\lambda}{(\lambda - t)^3} \quad &\Rightarrow \quad \mathrm{E}(X^2) = M''(0) = \frac{2}{\lambda^2} \\
&\vdots \\
M^{(k)}(t) &= \frac{k!\lambda}{(\lambda - t)^{k+1}} &\Rightarrow \quad \mathrm{E}(X^k) = M^{(k)}(0) = \frac{k!}{\lambda^k}.
\end{aligned}$$

Example: Calculating Moments Using MGF — $N(0,1)$

The MGF for $N(0,1)$ is

$$M(t) = e^{t^2/2}$$

The derivatives and the moments are thus

$$\begin{aligned}
M'(t) &= te^{t^2/2} \quad \Rightarrow \quad \mathrm{E}(X) = M'(0) = 0 \\
M''(t) &= e^{t^2/2} + t^2 e^{t^2/2} \quad \Rightarrow \quad \mathrm{E}(X^2) = M''(0) = 1 \\
M^{(3)}(t) &= 3te^{t^2/2} + t^3 e^{t^2/2} \Rightarrow \quad \mathrm{E}(X^3) = M^{(k)}(0) = 0 \\
M^{(4)}(t) &= (3 + 3t + 3t^2 + t^3)e^{t^2/2} \Rightarrow \quad \mathrm{E}(X^4) = M^{(k)}(0) = 3 \\
&\vdots
\end{aligned}$$

MGF for $a + bX$

If X has the MGF $M_X(t)$ and $Y = a + bX$, then the MGF for Y is

$$
\begin{aligned}
M_Y(t) &= \mathrm{E}(e^{tY}) \\
&= \mathrm{E}(e^{at+btX}) \\
&= \mathrm{E}(e^{at}e^{btX}) \\
&= e^{at}\,\mathrm{E}(e^{btX}) \\
&= e^{at}M_X(bt)
\end{aligned}
$$

MGF for $N(\mu, \sigma^2)$

If $X \sim N(0, 1)$, we know in L04 that

$$Y = \mu + \sigma X \sim N(\mu, \sigma^2).$$

As the MGF for X is known to be $M_X(t) = e^{t^2/2}$, we can obtain the MGF for $Y = \mu + \sigma X$ from $M_X(t)$ to be

$$M_Y(t) = e^{\mu t} M_X(\sigma t) = e^{\mu t} e^{\sigma^2 t^2/2} = e^{\mu t + \sigma^2 t^2/2}.$$

Cauchy Distribution Has No MGF

The Cauchy Distribution has the PDF

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty \le x < \infty.$$

Its MGF would be

$$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x$$
$$> \begin{cases} \int_0^{\infty} \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x = \infty & \text{since } \lim_{x\to\infty} \frac{e^{tx}}{\pi(1+x^2)} = \infty \text{ if } t > 0, \\ \int_{-\infty}^{0} \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x = \infty & \text{since } \lim_{x\to-\infty} \frac{e^{tx}}{\pi(1+x^2)} = \infty \text{ if } t < 0 \end{cases}$$

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$$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x$$
$$> \begin{cases} \int_0^\infty \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x = \infty & \text{since } \lim_{x\to\infty} \frac{e^{tx}}{\pi(1+x^2)} = \infty \text{ if } t > 0, \\ \int_{-\infty}^0 \frac{e^{tx}}{\pi(1+x^2)} \mathrm{d}x = \infty & \text{since } \lim_{x\to-\infty} \frac{e^{tx}}{\pi(1+x^2)} = \infty \text{ if } t < 0 \end{cases}$$

Remark

- If X has an MGF $M(t)$ that exists for t in an open interval containing 0, then all the moments $\mathrm{E}(X^k)$ exist.
- If X doesn't have all the moments $\mathrm{E}(X^k)$, then X has a **heavier tail** than those with all the moments.

MGFs for Common Discrete Distributions

Name and range	PMF at k	Mean	Variance	MGF
Bernoulli(p) on $\{0,1\}$	$\begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}$	p	$p(1-p)$	pe^t+1-p
Binomial(n,p) on $\{0,1,\ldots,n\}$	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	$(pe^t+1-p)^n$
Geometric(p) on $\{1,2,3\ldots\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Negative Binomial(r,p) on $\{r,r+1,r+2,\ldots\}$	$\binom{k-1}{r-1}p^r(1-p)^{k-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$
Poisson(λ) on $\{0,1,2,\ldots\}$	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ	$\exp(\lambda(e^t-1))$

- MGF for the Hypergeometric distribution exists but is complicated

MGFs for Common Continuous Distributions

Name	PDF $f(x)$	Range	Mean	Variance	MGF $M(t)$
Exponential(λ)	$\lambda e^{-\lambda x}$,	$0 \leq x < \infty$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda - t}$, $t < \lambda$
Gamma(α, λ)	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$,	$0 \leq x < \infty$	α/λ	α/λ^2	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha$, $t < \lambda$
Normal(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$,	$-\infty < x < \infty$	μ	σ^2	$\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$
Cauchy	$\frac{1}{\pi(1+x^2)}$,	$-\infty < x < \infty$	not exist	not exist	does not exist

- MGF for the Beta distribution exists but is complicated.

MGF for Sum of Independent R.V.'s

If X and Y are independent r.v.'s with MGF's $M_X(t)$ and $M_Y(t)$, then $M_{X+Y}(t) = M_X(t)M_Y(t)$ on the common interval where both MGF's exist.

Proof.

$$\begin{aligned} M_{X+Y}(t) = \mathrm{E}(e^{t(X+Y)}) &= \mathrm{E}(e^{tX}e^{tY}) \\ &= \mathrm{E}(e^{tX})\,\mathrm{E}(e^{tY}) \quad \text{since } X, Y \text{ are indep.} \\ &= M_X(t)M_Y(t) \end{aligned}$$

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More generally, if $X_1, \ldots, X_n$ are independent with corresponding MGF $M_{X_i}(t)$'s, then the MGF for $T = \sum_{i=1}^n X_i$ is

$$M_T(t) = \mathrm{E}[e^{t\sum_{i=1}^n X_i}] = \prod_{i=1}^n \mathrm{E}[e^{tX_i}] = \prod_{i=1}^n M_{X_i}(t),$$

and it exists on the common interval where all MGF's exist.

The MGF Uniquely Determines the Distribution (★★★★★)

If the moment-generating function $M(t)$ exists for t in an open interval containing 0, like $(-t_0, t_0)$, for some $t_0 > 0$, then it uniquely determines the probability distribution.

The MGF Uniquely Determines the Distribution (★★★★★)

If the moment-generating function $M(t)$ exists for t in an open interval containing 0, like $(-t_0, t_0)$, for some $t_0 > 0$, then it uniquely determines the probability distribution.

That is, if X and Y have identical MGF

$$M_X(t) = M_Y(t) \quad \text{for all } t \text{ in an open interval containing } 0,$$

then X and Y have the same probability distribution.

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then X and Y have the same probability distribution.

- **Ex1** If X has the MGF $M(t) = (1/2)^{10}(e^t + 1)^{10}$, then X must be $\sim \text{Bin}(n = 10, p = 1/2)$.
- **Ex2** If X has the MGF $M(t) = \exp(3(e^t - 1))$, what's the distribution of X?

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- **Ex1** If X has the MGF $M(t) = (1/2)^{10}(e^t + 1)^{10}$, then X must be $\sim \text{Bin}(n = 10, p = 1/2)$.
- **Ex2** If X has the MGF $M(t) = \exp(3(e^t - 1))$, what's the distribution of X? Poisson($\lambda = 3$)

Finding the Distribution of the Sum of Indep. R.V.'s Using MGF

Ex1. $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ are independent, what's the distribution of $X + Y$?

Finding the Distribution of the Sum of Indep. R.V.'s Using MGF

Ex1. $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ are independent, what's the distribution of $X + Y$?

Sol. The MGF for $X + Y$ is

$$M(t) = M_X(t)M_Y(t) = \exp\left(\mu_x t + \frac{\sigma_x^2 t^2}{2}\right) \exp\left(\mu_y t + \frac{\sigma_y^2 t^2}{2}\right)$$
$$= \exp\left((\mu_x + \mu_y)t + \frac{(\sigma_x^2 + \sigma_y^2)t^2}{2}\right)$$

which is the MGF for $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$, meaning

$$X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2).$$

Ex2. What's the distribution of $\sum_{i=1}^{n} X_i$ for independent $X_i \sim$ Poisson(λ_i)?

Finding the Distribution of the Sum of Indep. R.V.'s Using MGF

Ex2. What's the distribution of $\sum_{i=1}^{n} X_i$ for independent $X_i \sim$ Poisson(λ_i)?

Sol. The MGF for X_i is $M_{X_i}(t) = \exp(\lambda_i(e^t - 1))$. The MGF for $\sum_{i=1}^{n} X_i$ is

$$M(t) = \prod_{i=1}^{n} \exp(\lambda_i(e^t - 1)) = \exp\left((e^t - 1)\sum_{i=1}^{n} \lambda_i\right),$$

which is the MGF for Poisson$(\sum_{i=1}^{n} \lambda_i)$, meaning

$$\sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right).$$

Sum of i.i.d. Random Variables

If $X_1, X_2, \ldots, X_n$ are i.i.d. with MGF $M_X(t)$, the MGF for $\sum_{i=1}^{n} X_i$ would be

$$M(t) = \mathrm{E}[e^{t\sum_{i=1}^{n} X_i}] = \prod_{i=1}^{n} \mathrm{E}[e^{tX_i}] = \prod_{i=1}^{n} M_X(t) = (M_X(t))^n\,.$$

Sum of i.i.d. Random Variables

If $X_1, X_2, \ldots, X_n$ are i.i.d. with MGF $M_X(t)$, the MGF for $\sum_{i=1}^n X_i$ would be

$$M(t) = \mathsf{E}[e^{t\sum_{i=1}^n X_i}] = \prod_{i=1}^n \mathsf{E}[e^{tX_i}] = \prod_{i=1}^n M_X(t) = (M_X(t))^n \,.$$

Ex3. If $X_1, X_2, \ldots, X_n$ are i.i.d. Exponential(λ), what's the distribution of $\sum_{i=1}^n X_i$?

Sum of i.i.d. Random Variables

If $X_1, X_2, \ldots, X_n$ are i.i.d. with MGF $M_X(t)$, the MGF for $\sum_{i=1}^n X_i$ would be

$$M(t) = \mathrm{E}[e^{t\sum_{i=1}^n X_i}] = \prod_{i=1}^n \mathrm{E}[e^{tX_i}] = \prod_{i=1}^n M_X(t) = (M_X(t))^n \,.$$

Ex3. If $X_1, X_2, \ldots, X_n$ are i.i.d. Exponential(λ), what's the distribution of $\sum_{i=1}^n X_i$?

Sol. The MGF for Exponential(λ) is $M_X(t) = \dfrac{\lambda}{\lambda - t}$.
The MGF for $\sum_{i=1}^n X_i$ would be

$$M(t) = (M_X(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n,$$

which is the MGF for Gamma($\alpha = n, \lambda$), meaning

$$\sum_{i=1}^n X_i \sim \text{Gamma}(\alpha = n, \lambda).$$

Sum of i.i.d. Random Variables

Ex4. If $X_1, X_2, \ldots, X_n$ are i.i.d. Geometric(p), what's the distribution of $\sum_{i=1}^{n} X_i$?

Sum of i.i.d. Random Variables

Ex4. If $X_1, X_2, \ldots, X_n$ are i.i.d. Geometric(p), what's the distribution of $\sum_{i=1}^{n} X_i$?

Sol. The MGF for Geometric(p) is $M_X(t) = \dfrac{pe^t}{1-(1-p)e^t}$.
The MGF for $\sum_{i=1}^{n} X_i$ would be

$$M(t) = (M_X(t))^n = \left(\frac{pe^t}{1-(1-p)e^t}\right)^n,$$

which is the MGF for NegBin($r = n, p$), meaning

$$\sum_{i=1}^{n} X_i \sim \text{NegBin}(r = n, p).$$

Joint Moment Generating Functions (Joint MGF's)

For any n random variables $X_1, \ldots, X_n$, the *joint moment generating function (joint MGF)* is defined to be

$$M(t_1, \ldots, t_n) = E[e^{t_1 X_1 + \cdots + t_n X_n}].$$

- The MGF for an individual X_i can be obtained from the joint PDF by letting all but t_i be 0. That is,

$$M_{X_i}(t) = \mathrm{E}[e^{tX_i}] = M(0, \ldots, 0, t, 0, \ldots, 0)$$

  where the t is in the ith place.
- The joint MGF uniquely determines the joint distribution of $X_1, \ldots, X_n$ (★★★★★, proof too advanced for STAT 244)
- **Corollary**: $X_1, \ldots, X_n$ are independent if and only if their joint MGF is the product of their marginal MGF:

$$M(t_1, \ldots, t_n) = M_{X_1}(t_1) \ldots M_{X_n}(t_n).$$

Example: Proof of Independence by Joint MGF — Normal

Let X and Y be i.i.d. $N(\mu, \sigma^2)$. Prove that $X + Y$ and $X - Y$ are independent by computing their joint MGF.

Proof. The joint MGF for $X + Y$ and $X - Y$ is

$$\begin{aligned}
M(s,t) &= \mathrm{E}(e^{s(X+Y)+t(X-Y)}) \quad \text{(by definition)} \\
&= \mathrm{E}(e^{(s+t)X+(s-t)Y}) \\
&= \mathrm{E}(e^{(s+t)X})\,\mathrm{E}(e^{(s-t)Y}) \quad \text{(by indep of } X, Y) \\
&= M_X(s+t)M_Y(s-t) \\
&= \exp\left(\mu(s+t) + \frac{\sigma^2(s+t)^2}{2}\right)\exp\left(\mu(s-t) + \frac{\sigma^2(s-t)^2}{2}\right) \\
&= \underbrace{\exp\left(2\mu s + \sigma^2 s^2\right)}_{\text{MGF for } N(2\mu, 2\sigma^2)} \; \underbrace{\exp\left(\sigma^2 t^2\right)}_{\text{MGF for } N(0, 2\sigma^2)}
\end{aligned}$$

This shows

- $X + Y \sim N(2\mu, 2\sigma^2)$ and $X - Y \sim N(0, 2\sigma^2)$
- $X + Y$ and $X - Y$ are independent

Characteristic Functions

- Drawback of MGF: It may not exist.
- The *characteristic function* of a random variable X is defined to be

$$\phi(t) = \mathrm{E}(e^{itX}), \text{ where } i = \sqrt{-1} = \text{ the imaginary number.}$$

- $\phi(t)$ always exists since $|e^{it}| = 1$, even for Cauchy distribution.
- $\phi_X(t) = M_X(it)$ if $M_x(t)$ exists (See next page)
- $\phi_{a+bX}(t) = e^{ait}\phi_X(bt)$
- $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$ if X and Y are independent
- The characteristic function uniquely determines the distribution

Characteristic Functions for Common Distributions

Name and range	PMF at k	MGF	Characteristic Function
Binomial(n, p) on $\{0, 1, \ldots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	$(pe^t + 1 - p)^n$	$(pe^{it} + 1 - p)^n$
Geometric(p) on $\{1, 2, 3 \ldots\}$	$(1-p)^{k-1} p$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Negative Binomial(r, p) on $\{r, r+1, r+2, \ldots\}$	$\binom{k-1}{r-1} p^r (1-p)^{k-r}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$	$\left(\frac{pe^{it}}{1-(1-p)e^{it}}\right)^r$
Poisson(λ) on $\{0, 1, 2, \ldots\}$	$e^{-\lambda} \frac{\lambda^k}{k!}$	$\exp(\lambda(e^t - 1))$	$\exp(\lambda(e^{it} - 1))$

Name	PDF $f(x)$	Range	MGF	Characteristic Function
Exponential(λ)	$\lambda e^{-\lambda x}$,	$0 \leq x < \infty$	$\frac{\lambda}{\lambda - t}$	$\frac{\lambda}{\lambda - it}$
Gamma(α, λ)	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$,	$0 \leq x < \infty$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha$	$\left(\frac{\lambda}{\lambda - it}\right)^\alpha$
Normal(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$,	$-\infty < x < \infty$	$\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$	$\exp\left(\mu it - \frac{\sigma^2 t^2}{2}\right)$
Cauchy	$\frac{1}{\pi(1+x^2)}$,	$-\infty < x < \infty$	does not exist	$e^{-\lvert t \rvert}$