STAT 24400 Lecture 6
Section 3.5 Conditional Distributions

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Conditional Distributions of Discrete Random Variables

Example 1 — Gas Station (Revisit)

A gas station has both **self-service** and **full-service** islands, each with a single regular unleaded pump with 2 hoses.

X = the # of hoses in use on the self-service island, and

Y = the # of hoses in use on the full-service island

The joint PMF of X and Y:

		Y (full-service)		
	$p(x, y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $\mathrm{P}(Y = 1 \mid X = 2)$?

	$p(x,y)$	Y 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

	$p(x, y)$	Y 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

By the definition of conditional probability,

$$\mathrm{P}(Y = 1 \mid X = 2) = \frac{\mathrm{P}(X = 2, Y = 1)}{\mathrm{P}(X = 2)} = \frac{p(2, 1)}{p_X(2)} = \frac{0.14}{0.50} = 0.28.$$

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The conditional PMF of Y given $X=2$ is

$$p_{Y|X}(y \mid x=2) = \frac{\mathrm{P}(X=2, Y=y)}{\mathrm{P}(X=2)} = \frac{p(2,y)}{p_X(2)}$$

y	0	1	2
$p_{Y\|X}(y \mid x=2)$	$\frac{0.06}{0.50} = 0.12$	$\frac{0.14}{0.50} = 0.28$	$\frac{0.30}{0.50} = 0.60$

	$p(x,y)$	Y = 0	1	2	$p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

Similarly, the conditional PMF of Y given $X = 0$ is

$$p_{Y|X}(y \mid x = 0) = \frac{\mathrm{P}(X = 0, Y = y)}{\mathrm{P}(X = 0)} = \frac{p(0, y)}{p_X(0)}$$

y	0	1	2
$p_{Y\|X}(y \mid x = 0)$	$\frac{0.10}{0.16} = 0.625$	$\frac{0.04}{0.16} = 0.25$	$\frac{0.02}{0.16} = 0.125$

	$p(x,y)$	Y = 0	1	2	$p_X(x)$
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$p_{Y\|X}(y \mid x = 0)$	$\frac{0.10}{0.16} = 0.625$	$\frac{0.04}{0.16} = 0.25$	$\frac{0.02}{0.16} = 0.125$

and the conditional PMF of Y given $X = 1$ is

$$p_{Y|X}(y \mid x = 1) = \frac{\mathrm{P}(X = 1, Y = y)}{\mathrm{P}(X = 1)} = \frac{p(1, y)}{p_X(1)}$$

y	0	1	2
$p_{Y\|X}(y \mid x = 1)$	$\frac{0.08}{0.34} \approx 0.235$	$\frac{0.20}{0.34} \approx 0.588$	$\frac{0.06}{0.34} \approx 0.176$

Conditional Distributions

Suppose X & Y are two discrete r.v.'s with joint PMF $p(x, y)$ and marginal PMF's $p_X(x)$ and $p_Y(y)$ respectively

The **conditional PMF** for Y given $X = x$ is

$$p_{Y|X}(y \mid x) = \frac{\mathrm{P}(X = x, Y = y)}{\mathrm{P}(X = x)} = \frac{p(x, y)}{p_X(x)},$$

The **conditional PMF** for X given $Y = y$ is

$$p_{X|Y}(x \mid y) = \frac{\mathrm{P}(X = x, Y = y)}{\mathrm{P}(Y = y)} = \frac{p(x, y)}{p_Y(y)},$$

	$p(x, y)$	Y 0	1	2	$p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
	$p_Y(y)$	0.24	0.38	0.38	

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	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
	$p_Y(y)$	0.24	0.38	0.38	

	$p(y \mid x)$	Y 0	1	2	row sum
	0	$\frac{0.10}{0.16} = 0.625$	$\frac{0.04}{0.16} = 0.25$	$\frac{0.02}{0.16} = 0.125$	1
X	1	$\frac{0.08}{0.34} \approx 0.235$	$\frac{0.20}{0.34} \approx 0.588$	$\frac{0.06}{0.34} \approx 0.176$	1
	2	$\frac{0.06}{0.50} = 0.12$	$\frac{0.14}{0.50} = 0.28$	$\frac{0.30}{0.50} = 0.6$	1
	$p_Y(y)$	0.24	0.38	0.38	

- Each row is a PMF for Y given some x value
- Observed the row sums of $p_{Y|X}(y \mid x)$ are all 1

	$p(x,y)$	Y = 0	1	2	$p_X(x)$
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
	$p_Y(y)$	0.24	0.38	0.38	

	$p(x \mid y)$	Y = 0	1	2	$p_X(x)$
X	0	$\frac{0.10}{0.24} \approx 0.417$	$\frac{0.04}{0.38} \approx 0.105$	$\frac{0.02}{0.38} \approx 0.053$	0.16
	1	$\frac{0.08}{0.24} \approx 0.333$	$\frac{0.20}{0.38} \approx 0.526$	$\frac{0.06}{0.38} \approx 0.158$	0.34
	2	$\frac{0.06}{0.24} = 0.25$	$\frac{0.14}{0.38} \approx 0.368$	$\frac{0.30}{0.38} \approx 0.790$	0.50
column sum		1	1	1	

- Each column is a PMF for X given some y value
- Observed the column sums $p_{X|Y}(x \mid y)$ are all 1

In summary,

A conditional PMF of Y given $X = x$ is $p_{Y|X}(y \mid x) = \dfrac{p(x, y)}{p_X(x)}$ which satisfies

$$0 \leq p_{Y|X}(y \mid x) \leq 1 \quad \text{and} \quad \sum_y p_{Y|X}(y \mid x) = 1, \quad \text{for all } x.$$

A conditional PMF of X given $Y = y$ is $p_{X|Y}(x \mid y) = \dfrac{p(x, y)}{p_Y(y)}$ which satisfies

$$0 \leq p_{X|Y}(x \mid y) \leq 1 \quad \text{and} \quad \sum_x p_{X|Y}(x \mid y) = 1, \quad \text{for all } y.$$

Example 2 — Poisson

For independent r.v.'s $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$, recall in L05, we show that

$$T = X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2).$$

Q: Given $T = X_1 + X_2 = t$, what's the conditional PMF of X_1?

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Q: Given $T = X_1 + X_2 = t$, what's the conditional PMF of X_1?

$$\begin{aligned} \mathrm{P}(X_1 = x \mid T = t) &= \frac{\mathrm{P}(\{X_1 = x\} \cap \{T = t\})}{\mathrm{P}(T = t)} \\ &= \frac{\mathrm{P}(\{X_1 = x\} \cap \{X_2 = t - x\})}{\mathrm{P}(T = t)} \\ &= \frac{e^{-\lambda_1}\lambda_1^x/x! \cdot e^{-\lambda_2}\lambda_2^{t-x}/(t-x)!}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^t/t!} \\ &= \binom{t}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{t-x}, \ 0 \le x \le t. \end{aligned}$$

i.e., given $X_1 + X_2 = t$, $X_1 \sim \text{Bin}(t, \frac{\lambda_1}{\lambda_1+\lambda_2})$.

Conditional Distributions of Continuous Random Variables

Conditional Distributions of Continuous Random Variables

Suppose X & Y are two discrete r.v.'s with joint PDF $f(x, y)$ and marginal PMF's $f_X(x)$ and $f_Y(y)$ respectively.

The **conditional PDF** for Y given $X = x$ is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}$$

The **conditional PDF** for X given $Y = y$ is

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

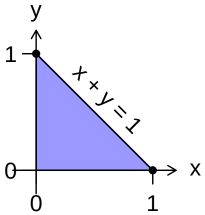
Example 3 — Deluxe Mixed Nuts

Recall in Lecture 5, the joint PDF for

$$X = \text{ the weight of almonds, and } Y = \text{ the weight of cashews}$$

in a can of mixed nuts is

$$f(x, y) = \begin{cases} 24xy & \text{if } 0 \leq x, y \leq 1, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$



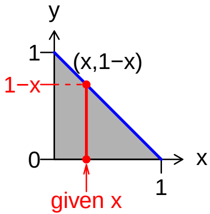
We calculated in L05 the marginal PDF's for X and for Y:

$$f_X(x) = 12x(1-x)^2, \quad f_Y(y) = 12y(1-y)^2, \text{ for } 0 \leq x, y \leq 1.$$

Conditional Distributions — Mixed Nuts

The conditional PDF $f_{X|Y}(x \mid y)$ of X (almond) given $Y = y$ (cashew) is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}$$



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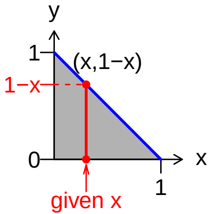

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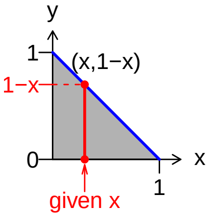
$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{24xy}{12y(1-y)^2} \\
&= \frac{2x}{(1-y)^2}, \qquad \text{for } 0 \le x \le 1-y\,.
\end{aligned}
$$



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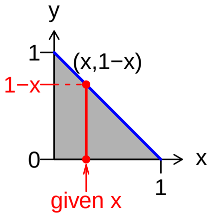
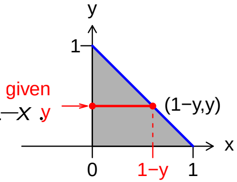


Similarly, the conditional PDF $f_{Y|X}(y \mid x)$ of Y (cashew) given $X = x$ (almond) is

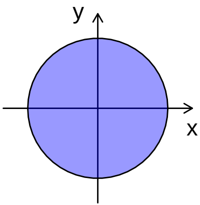
$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \quad \text{for } 0 \le y \le 1-x\,.$$



Example 4: Uniform Disk

(X, Y) is chosen uniformly at random from the unit disk, $\{x^2 + y^2 \leq 1\}$. The joint PDF is

$$f(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1, \\ 0, & \text{otherwise} \end{cases}$$



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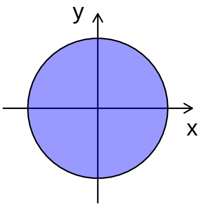


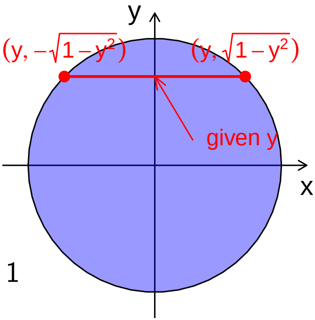
$$f(x,y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Marginal PMF of Y

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\mathrm{d}x$$

$$= \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\mathrm{d}x$$

$$= \frac{2}{\pi}\sqrt{1-y^2} \quad \text{for } -1 \leq y \leq 1$$



The PMF for X is the same $f_X(x) = \frac{2}{\pi}\sqrt{1-y^2}$ for $-1 \leq y \leq 1$.

Example 4: Uniform Disk (2)

The Conditional PDF for X given $Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-y^2}}{\pi}} = \frac{1}{2\sqrt{1-y^2}}$$

for $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$, which is *constant in x*.

In other words, given $Y = y$,

$$X \sim \text{Uniform}[-\sqrt{1-y^2}, \sqrt{1-y^2}].$$

Likewise, given $X = x$, Y is Uniform on $[-\sqrt{1-x^2}, \sqrt{1-x^2}]$

Note that

- the *marginal* PDF of Y (or of X) is not uniform, but
- the *conditional* PDF of $X|Y$ (or of $Y|X$) is uniform.

Example 5

Recall on p.40 of L05 slides, for X and Y w/ the joint PDF

$$f(x,y) = 6xy^2, \quad \text{for } 0 \le x, y \le 1,$$

We found the marginal PDF's of X and of Y to be

$$f_X(x) = 2x,\ 0 < x < 1, \quad \text{and} \quad f_Y(y) = 3y^2,\ 0 < y < 1.$$

The conditional PDF of y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6xy^2}{2x} = 3y^2, \quad 0 < y < 1.$$

which is exactly the *marginal PDF* of Y.

Recall in L05, we said X and Y are **independent** since $f(x,y) = 6xy^2 = (2x)(3y^2) = f_X(x)f_Y(y)$ for all $0 \le x, y \le 1$ and $f(x,y) = 0 = f_X(x)f_Y(y)$ elsewhere.

Conditional = Marginal, when Independent

What is the conditional distribution of Y given $X = x$ if X and Y are independent?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y).$$

i.e., *conditional PDF $Y|X$ is the marginal PDF of Y*.

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i.e., *conditional PDF $Y|X$ is the marginal PDF of Y*.

In fact, the following three are equivalent definitions of the independence of X and Y

- $f(x, y) = f_X(x)f_Y(y)$ ....(joint = product of marginal)
- $f_{Y|X}(y|X = x) = f_Y(y)$ ..(conditional $Y|X$ = marginal of Y)
- $f_{X|Y}(x|Y = y) = f_X(x)$ ..(conditional $X|Y$ = marginal of X)

Conditional = Marginal, when Independent

What is the conditional distribution of Y given $X = x$ if X and Y are independent?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y).$$

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- $f(x, y) = f_X(x)f_Y(y)$ ....(joint = product of marginal)
- $f_{Y|X}(y|X = x) = f_Y(y)$ ..(conditional $Y|X$ = marginal of Y)
- $f_{X|Y}(x|Y = y) = f_X(x)$ ..(conditional $X|Y$ = marginal of X)

All the things above apply to joint/conditional/marginal **PMF** for discrete X, Y., too.