STAT 24400 Lecture 5
Section 3.1-3.3 Joint & Marginal Distributions
Section 3.4 Independent Random Variables

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Why Consider Two or More Random Variables?

- Our focus so far has been on the distribution of a single random variable.
- Many situations involve two or more variables, for example,
  - counts of several species in ecological studies ($X_1 =$ count of deers, $X_2 =$ count of wolves, etc)
  - the x, y, and z components of wind velocity in atmospheric studies
- As the variables are often **correlated**, we need to consider them **jointly**, not separately

Joint Probability Distributions for Discrete R.V.

Joint Distribution of Two Discrete Random Variables

The *joint probability mass function (joint PMF)*, or, simply the *joint distribution*, for discrete r.v. $X_1, X_2, \ldots, X_k$ is defined as

$$\begin{aligned} p(x_1, x_2, \ldots, x_k) &= \mathrm{P}(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k). \\ &= \mathrm{P}(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \ldots \cap \{X_k = x_k\}) \end{aligned}$$

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Properties of joint PMF:

1. $p(x_1, x_2, \ldots, x_k) \geq 0$.
2. Define the probability for an event A as,

$$\mathrm{P}(A) = \mathrm{P}((x_1, x_2, \ldots, x_k) \in A) = \sum_{(x_1, x_2, \ldots, x_k) \in A} p(x_1, x_2, \ldots, x_k)\,.$$

3. If we set $A = \Omega$ (sample space) in (2), then

$$\mathrm{P}(\Omega) = \sum_{x_1, x_2, \ldots, x_k} p(x_1, x_2, \ldots, x_k) = 1\,.$$

Example 1 — Gas Station

A gas station has both **self-service** and **full-service** islands, each with a single regular unleaded pump with 2 hoses.

X = the # of hoses in use on the self-service island, and

Y = the # of hoses in use on the full-service island

The joint PMF of X and Y:

		Y (full-service)		
	$p(x,y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $\mathrm{P}(X = 2 \text{ and } Y = 1)$?

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What is $\mathrm{P}(X = 2 \text{ and } Y = 1)$? $p(2, 1) = 0.14$

Example 1 — Gas Station (2)

	$p(x, y)$	Y (full-service) 0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
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What is $\mathrm{P}(X + Y \leq 1)$?

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What is $\mathrm{P}(X + Y \leq 1)$?

$$\begin{aligned} &\mathrm{P}(X = 0, Y = 0) + \mathrm{P}(X = 0, Y = 1) + \mathrm{P}(X = 1, Y = 0) \\ &= p(0,0) + p(0,1) + p(1,0) \\ &= 0.10 + 0.04 + 0.08 = 0.22 \end{aligned}$$

Example 1 — Gas Station (3)

	$p(x,y)$	Y (full-service) 0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
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What is the probability that more self-service hoses in use than full service hoses $\mathrm{P}(X > Y)$?

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$$\begin{aligned} &\mathrm{P}(X = 1, Y = 0) + \mathrm{P}(X = 2, Y = 0) + \mathrm{P}(X = 2, Y = 1) \\ &= p(1, 0) + p(2, 0) + p(2, 1) \\ &= 0.08 + 0.06 + 0.14 = 0.28 \end{aligned}$$

Example 2 — Extended Hypergeometric Distributions

R red balls, B blue balls, G green balls

Suppose n balls are selected at random without replacement from the box above. Let

- X be the number of red balls obtained, and
- Y be the number of blue balls obtained.

The joint PMF of X and Y is

$$p(x,y) = \frac{\binom{R}{x}\binom{B}{y}\binom{G}{n-x-y}}{\binom{R+B+G}{n}}, \qquad \begin{array}{l} 0 \le x \le R \\ 0 \le y \le B \\ 0 \le n-x-y \le G \end{array}$$

If $R = 1$, $B = 2 = G = 2$, the joint PMF for (X, Y) for $n = 2$ draws is

	$p(x,y)$	Y = 0	1	2
X	0	1/10	4/10	1/10
	1	2/10	2/10	0

Example 3 — Coin & Die

Consider the game that, you toss a coin & roll a die at each round. If the coin lands heads, you win a prize, otherwise you win nothing. If the die shows a 1, then you stop playing, otherwise you continue. Find the joint PMF for X and Y below.

$$X = \text{the } \# \text{ of rounds you play, and } Y = \text{of times you win.}$$

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$$X = \text{the \# of rounds you play, and } Y = \text{of times you win.}$$

Sol. Observe $X \sim \text{Geometric}(1/6)$ since $X = \#$ of rolls needed to get the first $\boxed{\bullet}$. Given $X = x$, $Y \sim \text{Bin}(x, 1/2)$.

The joint PMF is thus

$$p(x,y) = \mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\mathrm{P}(Y = y \mid X = x)$$
$$= \underbrace{\left(\frac{5}{6}\right)^{x-1} \frac{1}{6}}_{\text{from Geom. distrib.}} \cdot \underbrace{\binom{x}{y} \left(\frac{1}{2}\right)^{y} \left(\frac{1}{2}\right)^{x-y}}_{\text{from Binomial distrib.}}$$

for $1 \leq x < \infty$ and $0 \leq y \leq x$.

Marginal Distribution

Obtaining PMF of X From the Joint Distribution of (X, Y)

	$p(x, y)$	Y = 0	1	2	Row Sum
X	0	0.10	0.04	0.02	
	1	0.08	0.20	0.06	
	2	0.06	0.14	0.30	

$\mathrm{P}(X = 0) =$

Obtaining PMF of X From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2	Row Sum
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	
	2	0.06	0.14	0.30	

$$\begin{aligned}\mathrm{P}(X = 0) &= \mathrm{P}(X = 0, Y = 0) + \mathrm{P}(X = 0, Y = 1) + \mathrm{P}(X = 0, Y = 2) \\ &= 0.10 + 0.04 + 0.02 = 0.16\end{aligned}$$

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Likewise,

$$\mathrm{P}(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$$

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X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

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$$\mathrm{P}(X=1) = 0.08 + 0.20 + 0.06 = 0.34$$
$$\mathrm{P}(X=2) = 0.06 + 0.14 + 0.30 = 0.50$$

Obtaining PMF of X From the Joint Distribution of (X, Y)

	$p(x, y)$	Y = 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

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Likewise,

$$\mathrm{P}(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$$
$$\mathrm{P}(X = 2) = 0.06 + 0.14 + 0.30 = 0.50$$

The PMF $p_X(x)$ of X is thus

x	0	1	2
$p_X(x)$	0.16	0.34	0.50

Obtaining PMF of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum				

$\mathrm{P}(Y = 0) =$

Obtaining PMF of Y From the Joint Distribution of (X, Y)

	$p(x, y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum		0.24		

$$\begin{aligned} \mathrm{P}(Y = 0) &= \mathrm{P}(X = 0, Y = 0) + \mathrm{P}(X = 1, Y = 0) + \mathrm{P}(X = 2, Y = 0) \\ &= 0.10 + 0.08 + 0.06 = 0.24 \end{aligned}$$

Obtaining PMF of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
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	2	0.06	0.14	0.30
Column sum		0.24	0.38	

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Column sum		0.24	0.38	0.38

$$\begin{aligned} \mathrm{P}(Y=0) &= \mathrm{P}(X=0, Y=0) + \mathrm{P}(X=1, Y=0) + \mathrm{P}(X=2, Y=0) \\ &= 0.10 + 0.08 + 0.06 = 0.24 \end{aligned}$$

Likewise,

$$\begin{aligned} \mathrm{P}(Y=1) &= 0.04 + 0.20 + 0.14 = 0.38 \\ \mathrm{P}(Y=2) &= 0.02 + 0.06 + 0.30 = 0.38 \end{aligned}$$

Obtaining PMF of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum	$p_Y(y)$	0.24	0.38	0.38

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Likewise,

$$\mathrm{P}(Y=1) = 0.04 + 0.20 + 0.14 = 0.38$$
$$\mathrm{P}(Y=2) = 0.02 + 0.06 + 0.30 = 0.38$$

The PMF $p_Y(y)$ of Y is thus

y	0	1	2
$p_Y(y)$	0.24	0.38	0.38

Marginal Distribution

The **marginal probability mass functions (marginal PMF's)** of X and of Y are obtained by summing $p(x, y)$ over values of *the other variable*.

$$p_X(x) = \sum_y p(x, y), \quad p_Y(y) = \sum_x p(x, y).$$

Example: Gas Station

	$p(x, y)$	Y 0	1	2	Row Sum $p_X(x)$
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
Column sum	$p_Y(y)$	0.24	0.38	0.38	

We call them **marginal distributions** because they show up at the table margins when the joint distribution is written in a tabular form

Example 2 — Extended Hypergeometric — Marginal

For X = the # of red balls and Y = the # of blue balls obtained from drawing n **balls** at random w/o replacement from the box:

$$\boxed{R \text{ red balls}, \quad B \text{ blue balls}, \quad G \text{ green balls}},$$

recall the joint PMF of X and Y is

$$p(x,y) = \frac{\binom{R}{x}\binom{B}{y}\binom{G}{n-x-y}}{\binom{R+B+G}{n}}, \qquad \begin{array}{l} 0 \le x \le R, \\ 0 \le y \le B, \\ 0 \le n-x-y \le G. \end{array}$$

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The marginal PMF of X is $p_X(x) = \sum_y p(x,y)$

$$p_X(x) = \sum_y \frac{\binom{R}{x}\binom{B}{y}\binom{G}{n-x-y}}{\binom{R+B+G}{n}} = \frac{\binom{R}{x}\sum_y \binom{B}{y}\binom{G}{n-x-y}}{\binom{R+B+G}{n}} = \frac{\binom{R}{x}\binom{B+G}{n-x}}{\binom{R+B+G}{n}}.$$

where $\sum_y \binom{B}{y}\binom{G}{n-x-y} = \binom{B+G}{n-x}$ comes from the Vandermonde identity $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$. Thus X is hypergeometric.

Example 3 — Coin & Die — Marginal of Y

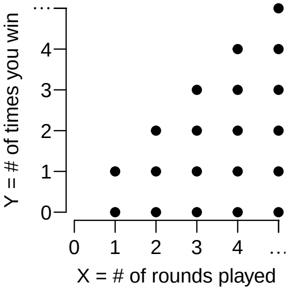
For the coin & dice game, recall the joint PMF for $X = \#$ of rounds played and $Y = \#$ of times you win is

$$p(x,y) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)\cdot\binom{x}{y}\left(\frac{1}{2}\right)^{y}\left(\frac{1}{2}\right)^{x-y}, \qquad \begin{array}{l} 1 \le x < \infty \\ 0 \le y \le x. \end{array}$$

Note the joint PMF is only defined at the black dots below

The marginal PMF for $Y = (\#$ of times you win$)$ is

$$p_Y(y) = \sum_{x=\max(1,y)}^{\infty}\left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)\binom{x}{y}\left(\frac{1}{2}\right)^{x}.$$



For $y = 0$,

$$p_Y(0) = \sum_{x=1}^{\infty} \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{6}\right) \left(\frac{1}{2}\right)^{x} = \frac{1}{12} \sum_{x=1}^{\infty} \left(\frac{5}{12}\right)^{x-1}$$

$$= \frac{1}{12} \frac{1}{(1 - 5/12)} = \frac{1}{7}.$$

For $y = 1, 2, 3, \ldots$,

$$p_Y(y) = \sum_{x=y}^{\infty} \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{6}\right) \binom{x}{y} \left(\frac{1}{2}\right)^{x} = \frac{1}{5} \left(\frac{5}{12}\right)^{y} \sum_{x=y}^{\infty} \binom{x}{y} \left(\frac{5}{12}\right)^{x-y}$$

$$= \frac{1}{5} \left(\frac{5}{12}\right)^{y} \frac{1}{(1 - 5/12)^{y+1}} = \frac{12}{35} \left(\frac{5}{7}\right)^{y}.$$

where $\sum_{x=y}^{\infty} \binom{x}{y} \left(\frac{5}{12}\right)^{x-y} = 1/(1 - 5/12)^{y+1}$ comes from the **Negative Binomial expansion** in L03 that

$$\sum_{k=m}^{\infty} \binom{k}{m} u^{k-m} = \frac{1}{(1 - u)^{m+1}}.$$

Example 5 — Sum of Independent Poisson R.V.'s

Suppose $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$ are independent. Find the PMF of $T = X_1 + X_2$.

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- Strategy: Find the joint PMF of X_1 and T, and then sum over X_1 to obtain the marginal PMF of T.
- To find the joint PMF of X_1 and Y, the key is to realize that $\{X_1 = x, T = t\}$ means $\{X_1 = x, X_2 = t - x\}$,

$$\begin{aligned}
p(x,t) &= \mathrm{P}(X_1 = x, T = t) \\
&= \mathrm{P}(X_1 = x, X_2 = t - x) \\
&= \mathrm{P}(X_1 = x)\mathrm{P}(X_2 = t - x) \text{ (by indep. of } X_1 \& X_2) \\
&= e^{-\lambda_1}\frac{\lambda_1^x}{x!}e^{-\lambda_2}\frac{\lambda_2^{t-x}}{(t-x)!}
\end{aligned}$$

for $0 \le x \le t < \infty$.

The joint PMF of X_1 and $T = X_1 + X_2$ is

$$p(x,t) = e^{-\lambda_1}\frac{\lambda_1^x}{x!}e^{-\lambda_2}\frac{\lambda_2^{t-x}}{(t-x)!}, \quad \text{for } 0 \le x \le t < \infty.$$

Summing over x, we get the marginal PMF of T

$$\begin{aligned} p_T(t) = \sum_{x=0}^{t} p(x,t) &= \sum_{x=0}^{t} e^{-\lambda_1}\frac{\lambda_1^x}{x!}e^{-\lambda_2}\frac{\lambda_2^{t-x}}{(t-x)!} \\ &= \frac{e^{-\lambda_1-\lambda_2}}{t!}\sum_{x=0}^{t}\frac{t!}{x!(t-x)!}\lambda_1^x\lambda_2^{t-x} \\ &= \frac{e^{-\lambda_1-\lambda_2}}{t!}\sum_{x=0}^{t}\binom{t}{x}\lambda_1^x\lambda_2^{t-x} \quad (*) \\ &= e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^t}{t!}. \end{aligned}$$

At the step $(*)$, $\sum_{x=0}^{t}\binom{t}{x}\lambda_1^x\lambda_2^{t-x} = (\lambda_1+\lambda_2)^t$ because of the Binomial expansion $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$.

This shows that $T = X_1 + X_2 \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Joint Distribution of Continuous Random Variables

Joint Distribution of Two Continuous Random Variables

Let X and Y be continuous rv. Then $f(x, y)$ is their *joint probability density function* or *joint PDF* for X and Y if for any two-dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

In particular, if A is the two-dimensional rectangle $\{a \le x \le b, c \le y \le d\}$, then

$$P[(X, Y) \in A] = \mathrm{P}(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Conditions for a joint PDF

- It must be nonnegative: $f(x, y) \ge 0$ for all x and y
- $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1$

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The weight of peanuts in the can is thus $(1 - X - Y)$

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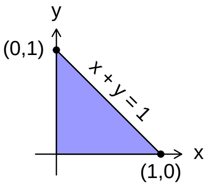
- Weights of the 3 types of nuts in a can are random but the **total** is **exactly 1 lb**
- In a randomly selected can, let

$$X = \text{ weight of almonds, and } Y = \text{ weight of cashews.}$$

  The weight of peanuts in the can is thus $(1 - X - Y)$
- Natural constraints on X & Y:

$$0 \leq X \leq 1, 0 \leq Y \leq 1, X + Y < 1$$



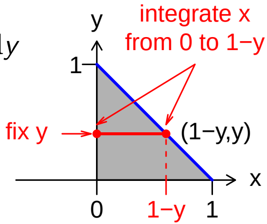
- Joint PDF of X & Y:

$$f(x, y) = \begin{cases} 24xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Checking Conditions on a Joint PDF

Clearly, $f(x, y) \geq 0$. It remains to check $\iint f(x, y)\mathrm{d}x\mathrm{d}y = 1$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} 24xy \, \mathrm{d}x \, \mathrm{d}y$$



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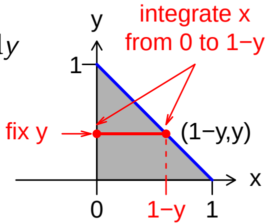


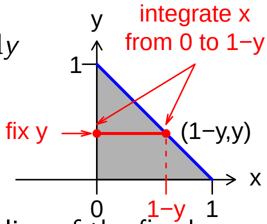
To compute the double integral above,

1. hold one variable fixed (e.g., y)

Checking Conditions on a Joint PDF

Clearly, $f(x,y) \geq 0$. It remains to check $\iint f(x,y)\mathrm{d}x\mathrm{d}y = 1$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y$$



To compute the double integral above,

1. hold one variable fixed (e.g., y)
2. integrate the other variable x along the line of the fixed y
   - key: express the end points of the line in terms of the fixed y, which will be the upper and lower limits for the integral over x

$$\int_0^{1-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=1-y} = 12(1-y)^2y$$

3. integrate the variable y that is fixed in the prior steps

$$\int_0^1\int_0^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 12(1-y)^2y\mathrm{d}y = 6y^2-8y^3+3y^4\Big|_0^1 = 1.$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X > 0.3)$

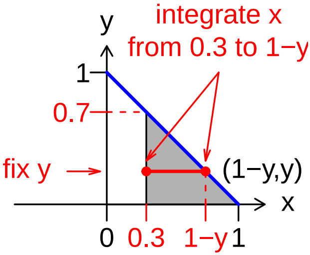
What is $\mathrm{P}(X > 0.3) = \mathrm{P}(\text{at least 30\% almonds in a can})$?

Finding Probabilities From the Joint PDF $\mathrm{P}(X > 0.3)$

What is $\mathrm{P}(X > 0.3) = \mathrm{P}(\text{at least 30\% almonds in a can})$?

$$\mathrm{P}(X > 0.3) = \iint_{x>0.3} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7}\int_{0.3}^{1-y} 24xy\,\mathrm{d}x\mathrm{d}y$$



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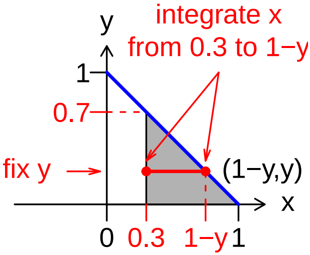

where

$$\int_{0.3}^{1-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0.3}^{x=1-y}$$

$$= 12((1-y)^2 - 0.3^2)y = 12(0.91y - 2y^2 + y^3).$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X > 0.3)$

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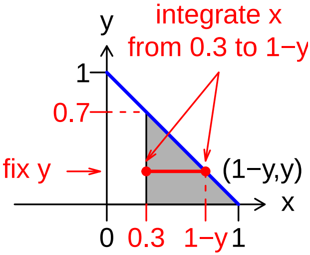

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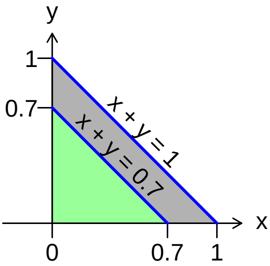
Putting it back to the double integral, we get

$$\int_0^{0.7}\int_{0.3}^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^{0.7} 12(0.91y - 2y^2 + y^3)\mathrm{d}y$$

$$= 5.46y^2 - 8y^3 + 3y^4\Big|_0^{0.7} = 0.6517.$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?

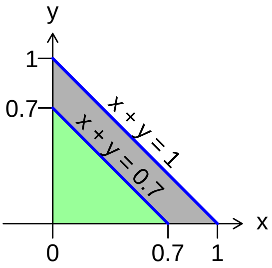


$$\mathrm{P}(\text{less than 30\% are Peanuts})$$
$$=$$
$$=$$
$$=$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?

$$
\begin{aligned}
&\mathrm{P}(\text{less than 30\% are Peanuts}) \\
=&\mathrm{P}(\text{at least 70\% are almonds or cashews}) \\
=& \\
=&
\end{aligned}
$$



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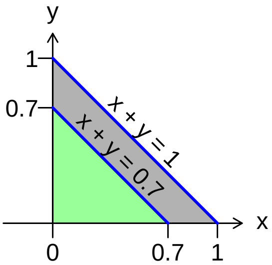
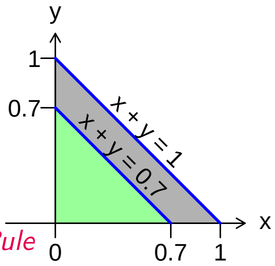


$$
\begin{aligned}
&\mathrm{P}(\text{less than 30\% are Peanuts}) \\
=&\,\mathrm{P}(\text{at least 70\% are almonds or cashews}) \\
=&\,\mathrm{P}(X + Y > 0.7) \\
=&
\end{aligned}
$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X+Y>0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?

$$\begin{aligned}&\mathrm{P}(\text{less than 30\% are Peanuts})\\=&\,\mathrm{P}(\text{at least 70\% are almonds or cashews})\\=&\,\mathrm{P}(X+Y>0.7)\\=&\,1-\mathrm{P}(X+Y\le 0.7)\quad\text{by } \textit{Complement Rule}\end{aligned}$$



Finding Probabilities From the Joint PDF $\mathrm{P}(X + Y > 0.7)$

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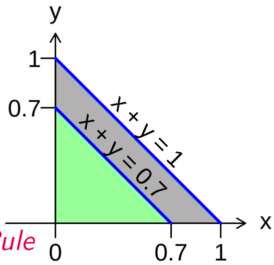


$$
\begin{aligned}
& \mathrm{P}(\text{less than 30\% are Peanuts}) \\
= & \mathrm{P}(\text{at least 70\% are almonds or cashews}) \\
= & \mathrm{P}(X + Y > 0.7) \\
= & 1 - \mathrm{P}(X + Y \leq 0.7) \quad \text{by } \textit{Complement Rule}
\end{aligned}
$$

where

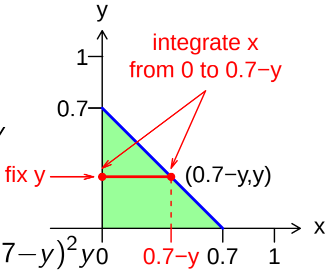
$\mathrm{P}(X + Y > 0.7) =$ integral of $f(x, y)$ over the **gray** region

$\mathrm{P}(X + Y < 0.7) =$ integral of $f(x, y)$ over the **green** region

Finding Probabilities From the Joint PDF $\mathrm{P}(X+Y>0.7)$ (Cont'd)

$$\mathrm{P}(X+Y<0.7)=\iint_{x+y<0.7} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\mathrm{d}y$$

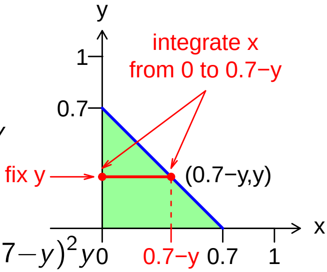


where

$$\int_0^{0.7-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=0.7-y} = 12(0.7-y)^2y$$

Finding Probabilities From the Joint PDF $\mathrm{P}(X+Y>0.7)$ (Cont'd)

$$\mathrm{P}(X+Y<0.7) = \iint_{x+y<0.7} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\mathrm{d}y$$



where

$$\int_0^{0.7-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=0.7-y} = 12(0.7-y)^2y$$

Putting it back to the double integral, we get

$$\begin{aligned}\int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{0.7} 12(0.7-y)^2y\mathrm{d}y = \int_0^{0.7}(-4y)\mathrm{d}(0.7-y)^3 \\ &= -4y(0.7-y)^3\Big|_0^{0.7} + \int_0^{0.7} 4(0.7-y)^3\mathrm{d}y \\ &= 0-(0.7-y)^4\Big|_0^{0.7} = (0.7)^4 = 0.2401.\end{aligned}$$

Hence, $\mathrm{P}(\text{less than } 30\% \text{ peanut}) = 1-0.2401 = 0.7599$.

Obtaining Marginal PDF's From Joint PDF

Given the joint PDF $f(x, y)$ of two continuous random variables, the *marginal probability density function (p)*, or simply the *marginal density*, of X and Y, can be obtained by *integrating the joint PDF over the other variable*.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y, \quad \text{for } -\infty < x < \infty,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x, \quad \text{for } -\infty < y < \infty.$$

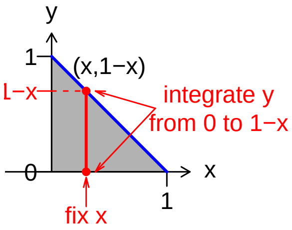
Recall the **marginal PMF's** of discrete random variables are obtained by *summing the joint PMF over values of the other variable*.

$$p_X(x) = \sum\nolimits_y p(x, y), \quad p_Y(y) = \sum\nolimits_x p(x, y).$$

Back to Example 6 (Deluxe Mixed Nuts)

The marginal PDFs of X (almond) is

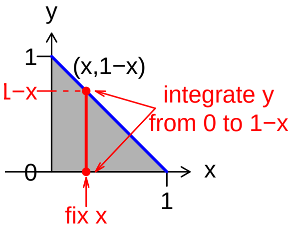
$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y \\
&= \int_0^{1-x} 24xy\mathrm{d}y = 12xy^2\Big|_{y=0}^{y=1-x} \\
&= 12x(1-x)^2, \text{ for } 0 \le x \le 1.
\end{aligned}
$$



Back to Example 6 (Deluxe Mixed Nuts)

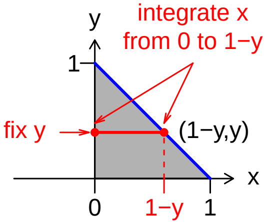
The marginal PDFs of X (almond) is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y$$
$$= \int_0^{1-x} 24xy\mathrm{d}y = 12xy^2\Big|_{y=0}^{y=1-x}$$
$$= 12x(1-x)^2, \text{ for } 0 \le x \le 1.$$



The marginal PDFs of Y (cashew) is

$$f_Y(y) = \int_{\infty}^{\infty} f(x,y)\mathrm{d}x$$
$$= \int_0^{1-y} 24xy\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=1-y}$$
$$= 12y(1-y)^2, \text{ for } 0 \le y \le 1\,.$$



Joint Cumulative Distribution Functions (Joint CDF)

Joint Cumulative Distribution Functions (Joint CDF)

The joint cumulative distribution function of the k random variables $X_1, X_2, \ldots, X_k$ is the function defined by

$$F(x_1, \ldots, x_k) = \mathrm{P}(X_1 \leq x_1, \ldots, X_k \leq x_k).$$

The random variables $X_1, X_2, \ldots, X_k$ can be discrete or continuous, or some be discrete and some be continuous.

Properties of Joint CDF for Two Random Variables

The joint CDF for any two random variables (X, Y) has the following properties

1. $\lim_{x \to -\infty} F(x, y) = F(-\infty, y) = 0$ for all y
2. $\lim_{y \to -\infty} F(x, y) = F(x, -\infty) = 0$ for all x
3. $\lim_{x \to \infty} \lim_{y \to \infty} F(x, y) = F(\infty, \infty) = 1$
4. **Right-continuous**:

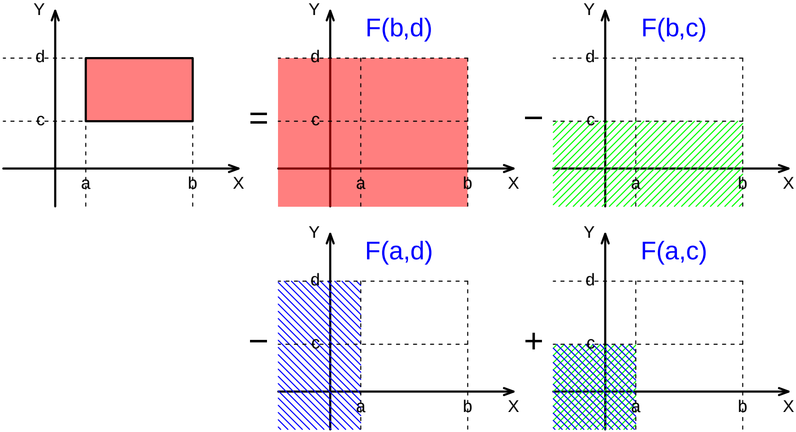
$$\lim_{h \to 0+} F(x + h, y) = \lim_{h \to 0+} F(x, y + h) = F(x, y)$$

   for all x and y

Properties of Joint CDF for Two Random Variables

5. **Non-decreasing**: For all $a < b$ and $c < d$,

$$\mathrm{P}(a < X \leq b, c < Y \leq d)$$
$$= F(b, d) - F(b, c) - F(a, d) + F(a, c) \geq 0$$



Joint PDF & CDF for Continuous R.V.'s

If $f(x, y)$ is the joint PDF for X and Y, their joint CDF is

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv.$$

Conversely, if $F(x, y)$ is the joint CDF for X and Y, their joint PDF is

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

Independent Random Variables

Independent Random Variables

- Recall that two events A and B are *independent* if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

- Two random variables X and Y are *independent* if

$$\mathrm{P}(X \in A, Y \in B) = \mathrm{P}(X \in A)\,\mathrm{P}(Y \in B)$$

  for any sets A and B.
- Two discrete random variables X and Y are *independent* if and only if

$$p(x, y) = p_X(x)p_Y(y) \quad \text{for all } x \text{ and } y,$$

  i.e., the joint PMF is the product of their marginal PMF's.

Independent Continous Random Variables

Two continuous random variables X and Y are *independent* if and only if

$$F(x,y) = F_X(x)F_Y(y) \quad \text{for all } x \text{ and } y,$$

i.e., the joint CDF is the product of their marginal CDF's.

Their joint PDF is

$$\begin{aligned} f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) &= F_X'(x)F_Y'(y) \\ &= f_X(x)f_Y(y). \end{aligned}$$

Conversely, if $f(x,y) = f_X(x)f_Y(y)$, their joint PDF is

$$\begin{aligned} F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(u)f_Y(v)dudv \\ &= \int_{-\infty}^{x} f_X(u)du \int_{-\infty}^{y} f_Y(v)dv = F_X(x)F_Y(y). \end{aligned}$$

i.e., independent if and only if joint PDF = Product of marginal PDF's

Are X and Y Independent?

	$p(x, y)$	y 1	2	3	
	1	0.05	0.10	0.05	
x	2	0.10	0.40	0.10	
	3	0.05	0.10	0.05	

Are X and Y Independent?

	$p(x,y)$	y = 1	2	3	$p_X(x)$
x	1	0.05	0.10	0.05	0.20
	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions

Are X and Y Independent?

	$p(x,y)$	y = 1	2	3	$p_X(x)$
	1	0.05	0.10	0.05	0.20
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	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions
2. Check whether

$$p(x,y) = p_X(x)p_Y(y)$$

Are X and Y Independent?

	$p(x,y)$	y = 1	2	3	$p_X(x)$
	1	0.05	0.10	0.05	0.20
x	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions
2. Check whether

$$p(x,y) = p_X(x)p_Y(y)$$

for all possible x, y **pairs.**

- $p(1,1) = 0.05 \neq 0.2 \times 0.2 = p_X(1)p_Y(1)$.
- X and Y are NOT independent.

Finding Joint PMF From Marginal PMF's When Independent

Given the marginal PMFs of two **independent** r.v.'s, X and Y, find their joint PMF.

	$p(x,y)$	y = 1	2	3	$p_X(x)$
x	1				0.2
	2				0.6
	3				0.2
	$p_Y(y)$	0.2	0.6	0.2	

Since X and Y are **independent**,

1. $p(1,1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
2. also $p(1,2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12$.
3. Repeat filling the blank for $p(x,y)$ by $p_X(x)p_Y(y)$ for all x, y pairs.

Finding Joint PMF From Marginal PMF's When Independent

Given the marginal PMFs of two **independent** r.v.'s, X and Y, find their joint PMF.

	$p(x,y)$	y = 1	2	3	$p_X(x)$
	1	0.04			0.2
x	2				0.6
	3				0.2
	$p_Y(y)$	0.2	0.6	0.2	

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Given the marginal PMFs of two **independent** r.v.'s, X and Y, find their joint PMF.

	$p(x, y)$	y = 1	2	3	$p_X(x)$
x	1	0.04	0.12		0.2
	2				0.6
	3				0.2
	$p_Y(y)$	0.2	0.6	0.2	

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Finding Joint PMF From Marginal PMF's When Independent

Given the marginal PMFs of two **independent** r.v.'s, X and Y, find their joint PMF.

	$p(x,y)$	$y=1$	$y=2$	$y=3$	$p_X(x)$
x	1	0.04	0.12	0.04	0.2
	2	0.12	0.36	0.12	0.6
	3	0.04	0.12	0.04	0.2
	$p_Y(y)$	0.2	0.6	0.2	

Since X and Y are **independent**,

1. $p(1,1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
2. also $p(1,2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12$.
3. Repeat filling the blank for $p(x,y)$ by $p_X(x)p_Y(y)$ for all x, y pairs.

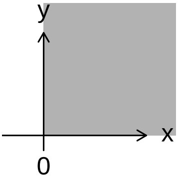
Finding Joint PDF From Marginal PDF's When Independent

Suppose the lifetimes X and Y of Batteries A and B are independent with PDFs

$$f_X(x) = e^{-x} \quad \text{and} \quad f_Y(y) = 2e^{-2y},$$

for $0 < x, y < \infty$, then their joint PDF is

$$f(x, y) = f_X(x) f_Y(y) = 2e^{-(x+2y)}, \quad 0 < x, y < \infty.$$



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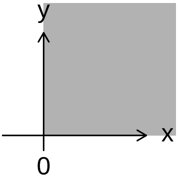


Question: $\mathrm{P}(X < Y) = \mathrm{P}(\text{Battery A dies before Battery B}) = ?$

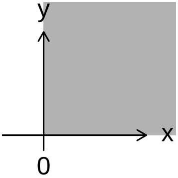
Finding Joint PDF From Marginal PDF's When Independent

Suppose the lifetimes X and Y of Batteries A and B are independent with PDFs

$$f_X(x) = e^{-x} \quad \text{and} \quad f_Y(y) = 2e^{-2y},$$

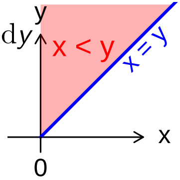
for $0 < x, y < \infty$, then their joint PDF is

$$f(x,y) = f_X(x)f_Y(y) = 2e^{-(x+2y)}, \quad 0 < x, y < \infty.$$



Question: $\mathrm{P}(X < Y) = \mathrm{P}(\text{Battery A dies before Battery B}) = ?$

$$\mathrm{P}(X < Y) = \iint_{x<y} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{0<x<y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y$$



Method 1

$$\iint_{0<x<y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\int_0^{y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y$$

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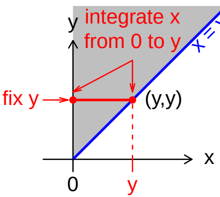

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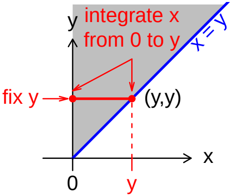

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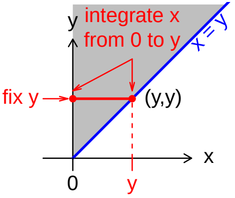

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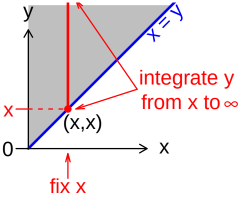

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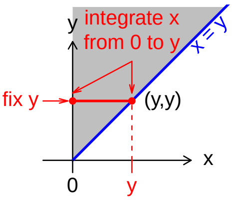

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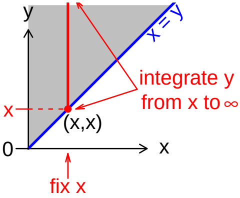

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Example — Are X & Y Independent?

Suppose the joint PDF of X, Y is

$$f(x, y) = 6xy^2, \quad \text{for } 0 \leq x, y \leq 1.$$

The marginal PDF of X is

$$f_X(x) = \int_0^1 6xy^2 \, \mathrm{d}y = 2xy^3 \Big|_{y=0}^{y=1} = 2x(1^3 - 0^3) = 2x, \quad 0 < x < 1.$$

The marginal PDF of Y is

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Are X and Y independent?

- Yes, since $f(x,y) = 6xy^2 = (2x)(3y^2) = f_X(x)f_Y(y)$ for all $0 \le x, y \le 1$ and $f(x,y) = 0 = f_X(x)f_Y(y)$ elsewhere.

A Simple Criterion for Checking Independence

So far, it seems like one must find the marginal distributions before checking independence. However, there is an easier way...

A Simple Criterion: X and Y are independent if the joint PMF/PDF can be written as the product of a function of x and a function of y.

$$f(x, y) = g(x)h(y), \quad \text{for all } x, y.$$

Here $g(x) \geq 0$ and $h(y) \geq 0$ are **not necessarily PMFs/PDFs**.

Are They Independent?

1. $p(x, y) = \dfrac{x+y}{36}$ for $x, y \in \{1, 2, 3\}$.
   - can't be factored, X and Y are NOT independent.

2. $p(x, y) = e^{-2}/(x!y!)$, for $x, y \in \{0, 1, 2, \ldots\}$.
   - factors into $g(x) = e^{-1}/x!$ and $h(y) = e^{-1}/y!$, so independent.

3. $f(x, y) = 8xy$ for $0 \leq x < y \leq 1$.
   - Does it factors into $g(x) = 8x$ and $h(y) = y$?

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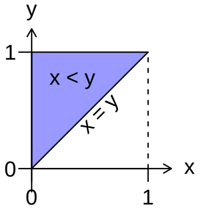
3. $f(x, y) = 8xy$ for $0 \le x < y \le 1$.
   - Does it factors into $g(x) = 8x$ and $h(y) = y$?
   - Watch out! When $x > y$,

$$f(x, y) = 0 \neq g(x)h(y).$$

   - X and Y are NOT independent.



Proof of the Simple Criterion for Independence

We prove the discrete case. The continuous case is similar. The marginal PDF of Y is

$$p_Y(y) = \sum\nolimits_x p(x,y) = \sum\nolimits_x g(x)h(y)$$
$$= h(y)\sum\nolimits_x g(x) = c_1 h(y),$$

in which c_1 is the constant $\sum_x g(x)$. Similarly, one can show $p_X(x) = c_2 g(x)$ where $c_2 = \sum_y h(y)$. Note that

$$c_1 c_2 = \sum\nolimits_x g(x) \sum\nolimits_y h(y) = \sum\nolimits_x \sum\nolimits_y g(x)h(y)$$
$$= \sum\nolimits_x \sum\nolimits_y f(x,y) = 1$$

since $p(x,y)$ is a joint PMF.

Thus $p_X(x)p_Y(x) = c_1 c_2 g(x)h(y) = g(x)h(y) = f(x,y)$.

Independence of Several Random Variables

- More generally, a sequence of random variables $X_1, X_2, \ldots, X_n$ are **(mutually) independent** if and only if

  $$\mathrm{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathrm{P}(X_1 \in A_1) \cdots \mathrm{P}(X_n \in A_n).$$

  for all sequence of events $A_1, A_2, \ldots$.
- Equivalently, the random variables $X_1, X_2, \ldots, X_n$ are **(mutually) independent** if and only if their joint distributions factors into the product of their marginal distributions.

  $$p(x_1, x_2, \ldots, x_n) = p_1(x_1)p_2(x_2)\ldots p_n(x_n) \qquad \text{for discrete rv's}$$
  $$f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_2)\ldots f_n(x_n) \qquad \text{for continuous rv's}$$

  for all $x_1, x_2, \ldots, x_n$.