STAT 234 Lecture 20
t Tests About a Population Mean
z-Tests About a Population Proportion
Section 9.2, 9.3, 9.4

Yibi Huang
Department of Statistics
University of Chicago

t-Tests about Means

One-sample t-Test of a Population Mean (Rejection Region)

Similar to the normal z test, the t-statistic for testing $H_0 : \mu = \mu_0$ when σ is unknown is

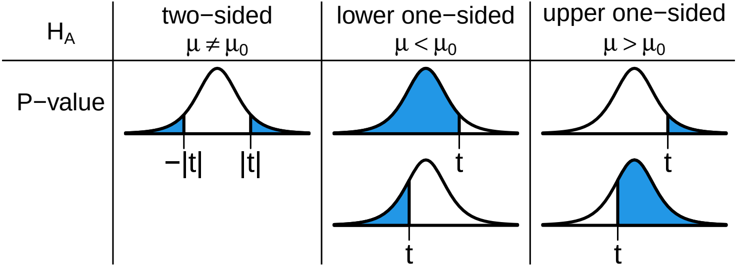
$$t = \frac{\overline{X} - \mu_0}{S / \sqrt{n}} \sim t_{n-1}$$

At the significance level α, we reject $H_0 : \mu = \mu_0$ if

- $t > t_{\alpha,n-1}$ for H_A: $\mu > \mu_0$
- $t < -t_{\alpha,n-1}$ for H_A: $\mu < \mu_0$
- $|t| > t_{\alpha/2,n-1}$ for H_A: $\mu \neq \mu_0$

One-sample t-Test of a Population Mean (P-value)

The calculation of the P-value also depends on H_A.



The bell-shape curve above is the t-curve with df $= n - 1$, not the normal curve. We reject H_0 when P-value $< \alpha$.

Example: Upper One-Sided Rejection Regions

Example 1: To test H_0: $\mu = 10$ v.s. H_A: $\mu > 10$,
t-stat $= 2.23$, sample size $n = 21$, df $= n - 1 = 20$.
The H_0 is rejected at significance level α if

$$t\text{-stat} > t_{\alpha,n-1} = \begin{cases} t_{0.1,20} \approx 1.325 & \text{for } \alpha = 0.1 \\ t_{0.05,20} \approx 1.725 & \text{for } \alpha = 0.05 \\ t_{0.01,20} \approx 2.528 & \text{for } \alpha = 0.01 \end{cases}$$

For the t-stat $= 2.23$,

- H_0 is rejected at levels $\alpha = 0.1$ and 0.05
- H_0 is NOT rejected at level $\alpha = 0.01$

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
	20	*1.325*	*1.725*	2.086	*2.528*	2.845	3.552	3.850
	21	1.323	1.721	2.080	2.518	2.831	3.527	3.819

```
qt(c(0.1,0.05,0.01), df=20, lower.tail=FALSE)
## [1] 1.325341 1.724718 2.527977
```

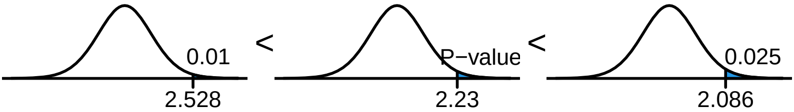
Finding Upper One-Sided P-Values in R

Example 1. H_0: $\mu = 10$ v.s. H_A: $\mu > 10$, t-stat $= 2.23$.

sample size $n = 21$,

df $= 21 - 1 = 20$

ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	*2.086*	*2.528*	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819



______ < one-sided P-value < _______

In R:

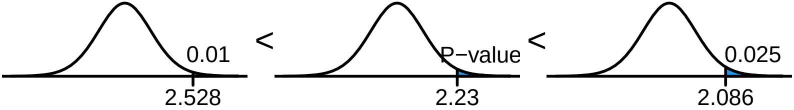
```r
pt(2.23,df=20,lower.tail=F)
## [1] 0.01868333
```

Finding Upper One-Sided P-Values in R

Example 1. H_0: $\mu = 10$ v.s. H_A: $\mu > 10$, t-stat $= 2.23$.

sample size $n = 21$,
df $= 21 - 1 = 20$

α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν 19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819



$\underline{0.01}$ < one-sided P-value < $\underline{0.025}$

In R:

```
pt(2.23,df=20,lower.tail=F)
## [1] 0.01868333
```

Example: Two-Sided Rejection Regions

Example 2: To test H_0: $\mu = 60$ v.s. H_A: $\mu \neq 60$,

t-stat $= 1.8$, sample size $n = 24$, df $= n - 1 = 23$.

The H_0 is rejected at significance level α if

$$|t\text{-stat}| > t_{\alpha/2,n-1} = \begin{cases} t_{0.05,23} \approx 1.714 & \text{for } \alpha = 0.1 \\ t_{0.025,23} \approx 2.069 & \text{for } \alpha = 0.05 \\ t_{0.005,23} \approx 2.807 & \text{for } \alpha = 0.01 \end{cases}$$

For the t-stat $= 2.23$,

- H_0 is rejected at level $\alpha = 0.1$
- H_0 is NOT rejected at levels $\alpha = 0.05$ and 0.01

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
	23	1.319	*1.714*	*2.069*	2.500	*2.807*	3.485	3.768
	24	1.318	1.711	2.064	2.492	2.797	3.467	3.745

```
qt(c(0.1,0.05,0.01)/2, df=23, lower.tail=FALSE)
## [1] 1.713872 2.068658 2.807336
```

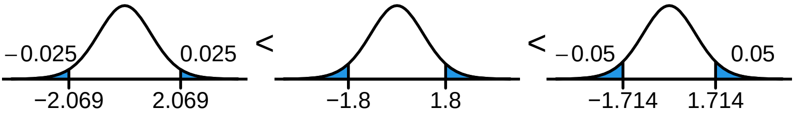
Finding Two-Sided P-Values

Example 2. H_0: $\mu = 60$ v.s. H_A: $\mu \neq 60$, t-stat $= 1.8$.

sample size $n = 24$

df $= n - 1 = 23$

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
	23	1.319	*1.714*	*2.069*	2.500	2.807	3.485	3.768
	24	1.318	1.711	2.064	2.492	2.797	3.467	3.745



________________ < two-sided P-value < ________________

In R:

```
2*pt(1.8,df=24-1,lower.tail=F)
## [1] 0.08499266
```

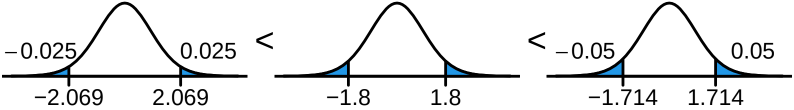
Finding Two-Sided P-Values

Example 2. H_0: $\mu = 60$ v.s. H_A: $\mu \neq 60$, t-stat $= 1.8$.

sample size $n = 24$

df $= n - 1 = 23$

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
	23	1.319	*1.714*	*2.069*	2.500	2.807	3.485	3.768
	24	1.318	1.711	2.064	2.492	2.797	3.467	3.745



$2 \times 0.025 = 0.05$ < two-sided P-value < $2 \times 0.05 = 0.1$

In R:

```
2*pt(1.8,df=24-1,lower.tail=F)
## [1] 0.08499266
```

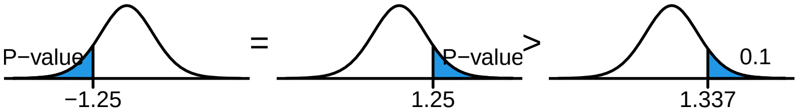
Finding Lower One-Sided P-Values

Example 3. H_0: $\mu = 20$ v.s. H_A: $\mu < 20$, t-stat $= -1.25$.

sample size $n = 17$

df $= 17 - 1 = 16$

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
	16	*1.337*	1.746	2.120	2.583	2.921	3.686	4.015
	17	1.333	1.740	2.110	2.567	2.898	3.646	3.965



one-sided P-value > _____

In R:

```
pt(-1.25, df=17-1)
## [1] 0.1146372
```

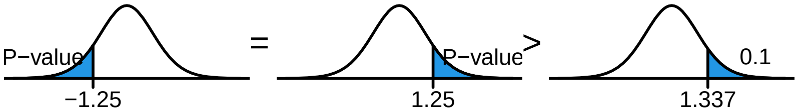
Finding Lower One-Sided P-Values

Example 3. H_0: $\mu = 20$ v.s. H_A: $\mu < 20$, t-stat $= -1.25$.

sample size $n = 17$

df $= 17 - 1 = 16$

α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν 15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	*1.337*	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965

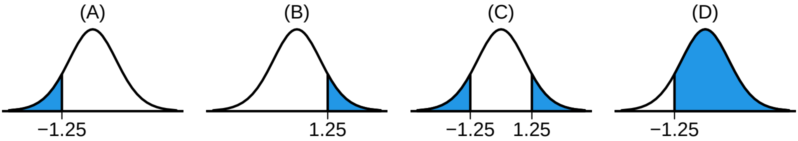


one-sided P-value > 0.1

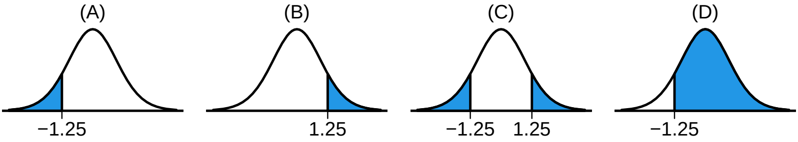
In R:

```
pt(-1.25, df=17-1)
## [1] 0.1146372
```

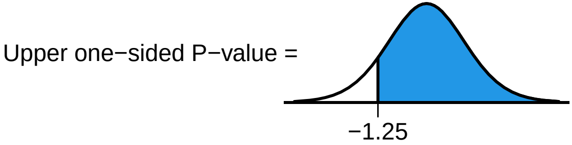
Example 4. For testing $H_0 : \mu = 20$ vs. $H_A : \mu > 20$ with a sample of size $n = 17$ and a t-statistic $= -1.25$, which of the following areas represents the correct P-value?



Example 4. For testing $H_0 : \mu = 20$ vs. $H_A : \mu > 20$ with a sample of size $n = 17$ and a t-statistic $= -1.25$, which of the following areas represents the correct P-value?



Ans: (D). Note $H_A : \mu > 20$ is upper one-sided. The P-value is the upper tail area above the t-statistic.




```
pt(-1.25, df=17-1, lower.tail=F)
## [1] 0.8853628
```

Cannot Find the Df on the t-Table?

Example 5. H_0: $\mu = 2.5$ v.s. H_A: $\mu > 2.5$, t-stat $= 2.8$.

sample size $n = 57$

df $= 57 - 1 = 56$

α		0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
	50	1.299	1.676	2.009	2.403	*2.678*	*3.261*	3.496
	60	1.296	1.671	2.000	2.390	*2.660*	*3.232*	3.460
	120	1.289	1.658	1.980	2.358	2.617	3.160	3.373

- df $= 57 - 1 = 56$ is not on the table.

Cannot Find the Df on the t-Table?

Example 5. H_0: $\mu = 2.5$ v.s. H_A: $\mu > 2.5$, t-stat $= 2.8$.

sample size $n = 57$

df $= 57 - 1 = 56$

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
	50	1.299	1.676	2.009	2.403	*2.678*	*3.261*	3.496
	60	1.296	1.671	2.000	2.390	*2.660*	*3.232*	3.460
	120	1.289	1.658	1.980	2.358	2.617	3.160	3.373

- df $= 57 - 1 = 56$ is not on the table.
- df $= 56$ is between df $= 50$ and 60 in the table

  If df $= 50$, P-value would be between 0.001 and 0.005.

  If df $= 60$, P-value would also be between 0.001 and 0.005.

Cannot Find the Df on the t-Table?

Example 5. H_0: $\mu = 2.5$ v.s. H_A: $\mu > 2.5$, t-stat $= 2.8$.

sample size $n = 57$

df $= 57 - 1 = 56$

ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	1.299	1.676	2.009	2.403	*2.678*	*3.261*	3.496
60	1.296	1.671	2.000	2.390	*2.660*	*3.232*	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373

- df $= 57 - 1 = 56$ is not on the table.
- df $= 56$ is between df $= 50$ and 60 in the table

  If df $= 50$, P-value would be between 0.001 and 0.005.

  If df $= 60$, P-value would also be between 0.001 and 0.005.
- So for df $= 56$, the P-value is also between 0.001 and 0.005

Example: Thermal Conductivity of Glass (Revisit)

Recall the Thermal Conductivity of Glass Example in L17.

Here are measurements of the thermal conductivity of 11 randomly selected pieces of a particular type of glass:

```
1.11, 1.07, 1.11, 1.07, 1.12, 1.08,
1.08, 1.18, 1.18, 1.18, 1.12
```

We want to investigate if the mean conductivity of this type of glass is greater than 1. The hypotheses are

$$\mathsf{H}_0\colon \mu = 1, \quad \mathsf{H}_A\colon \mu > 1.$$

where μ is the mean conductivity of this type of glass.

Example: Thermal Conductivity of Glass — t-Statistic

The sample mean and sample SD are

$$\overline{x} \approx 1.1182, \quad \text{and} \quad s \approx 0.04378.$$

```
conduct = c(1.11,1.07,1.11,1.07,1.12,1.08,1.08,
            1.18,1.18,1.18,1.12)
mean(conduct)
## [1] 1.118182
sd(conduct)
## [1] 0.04377629
```

The t-statistics is

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{1.118 - 1}{0.04378/\sqrt{11}} \approx 8.95,$$

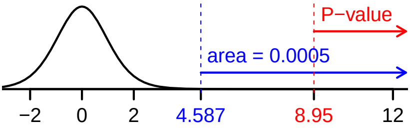
with $11 - 1 = 10$ degrees of freedom.

Here $\mu_0 = 1$ since $\mu = 1$ in H_0.

Example: Thermal Conductivity of Glass — P-value

Using t-table:

	α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν	9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
	10	1.372	1.812	2.228	2.764	3.169	4.144	*4.587*
	11	1.363	1.796	2.201	2.718	3.106	4.025	4.437



t-stat $= 8.95 > 4.587$

$\Rightarrow$ one sided P-value < 0.0005

In R:

```r
pt(8.95, df=10, lower.tail=F)
## [1] 2.175242e-06
```

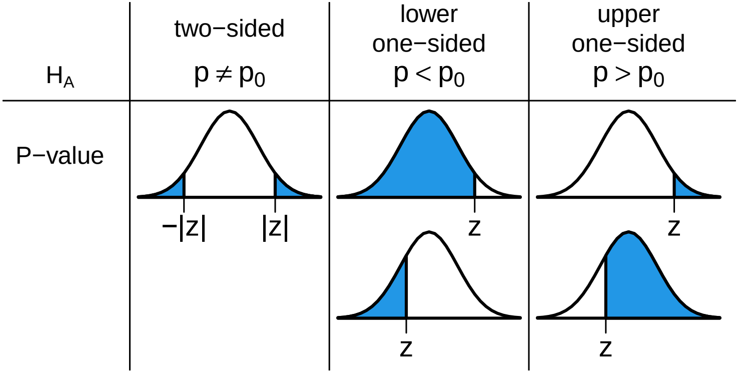
Conclusion: The tiny P-value 2.175×10^{-6} provides convincing evidence that the mean conductivity of this type of glass is > 1

Hypothesis Testing for a Proportion (Section 9.3)

Hypothesis Testing for a Proportion

Suppose we want to test $\text{H}_0 : p = p_0$ for some fixed value p_0.

Under H_0, $z = \dfrac{\widehat{p} - p_0}{\text{SE}} \mathrel{\dot\sim} N(0, 1)$, where $\text{SE} = \sqrt{\dfrac{p_0(1-p_0)}{n}}$



Here n should be so large that $np_0 \geq 10$, and $n(1-p_0) \geq 10$.

SE

Recall for confidence intervals, we use

$$\text{SE} = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$$

but for hypothesis testing we use

$$\text{SE} = \sqrt{\frac{p_0(1-p_0)}{n}}.$$

Why?

- Recall by CLT when n is large $\widehat{p} \dot{\sim} N\left(p, \sqrt{p(1-p)/n}\right)$
- When constructing CIs for p, p is unknown, so we estimate $\sqrt{p(1-p)/n}$ by $\sqrt{\widehat{p}(1-\widehat{p})/n}$
- Under $\text{H}_0 : p = p_0$, p is known to be p_0. There is no need to estimate p and the $\sqrt{p(1-p)/n}$ is simply $\sqrt{p_0(1-p_0)/n}$.

Example: Children's Play Preference (1)

An observational study was conducted at Chicago Children's Museum to determine the age at which a child's preferred play partner switched from gender-neutral to a same-sex peer

- For 5-year old children, 78 of 162 preferred to interact with a same-sex peer (48%)
- For 6-year old children, 59 of 97 preferred to interact with a same-sex peer (61%)

Under the *null hypothesis* of *no preference*, the probability that a child select a same-sex peer is $p = 0.5$

We want to test if 6-year old children had a preference interacting with a same-sex peer.

Example: Children's Play Preference (2)

We want to test $\text{H}_0 : p = 1/2$ versus $\text{H}_a : p \neq 1/2$

- Is the z test appropriate?
  Check whether $np_0 > 10$ and $n(1 - p_0) > 10$?
  (Yes; $np_0 = 97(0.5) = 48.5 > 10$ and $n(1 - p_0) = 48.5 > 10$)
- Test statistic

$$z = \frac{\widehat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} = \frac{0.61 - 0.5}{\sqrt{0.5(1 - 0.5)/97}} = \frac{0.11}{0.0508} = 2.17$$

- p-value is $2P(z > 2.17) = 2(0.015) = 0.03 < \alpha = 0.05$
- Conclude: 6-year old children prefer to interact with same-sex peers rather than gender-neutral

Conditions Required for Using the Large-Sample CI and Tests for Proportions

- The observations are (nearly) i.i.d. from the population studied.
  - If SRS, the sample size is at most 10% of the population size.
- The sample size n is large enough. A rule of thumb is that
  - to use the large-sample CI:
    $n\widehat{p}$ and $n(1-\widehat{p})$ need to be both ≥ 10
  - to use the large-sample test for the H_0 of $p = p_0$:
    np_0 and $n(1-p_0)$ need to be both ≥ 10