STAT 234 Lecture 16B
Overview of Confidence Intervals
Section 8.1

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Section 8.1 Overview of Confidence Intervals

- A plausible range of values for the population parameter is called a *confidence interval*.
- Using only a sample statistic (point estimate) to estimate a parameter is like fishing in a murky lake with a spear, and using a confidence interval is like fishing with a net.

We can throw a spear where we saw a fish but we will probably miss. If we toss a net in that area, we have a good chance of catching the fish.

- If we report a point estimate, we probably won't hit the exact population parameter. If we report a range of plausible values we have a good shot at capturing the parameter.

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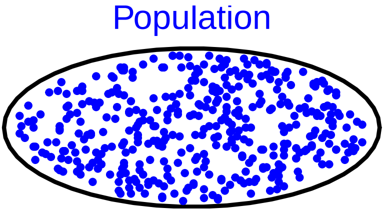
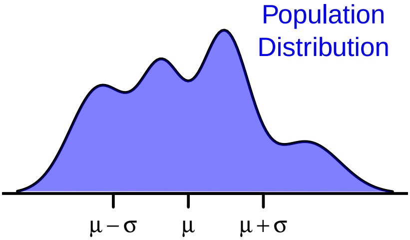
Variability in Estimation (Review)





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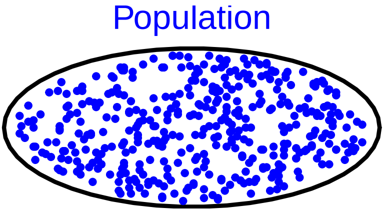

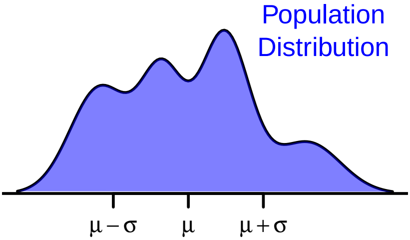

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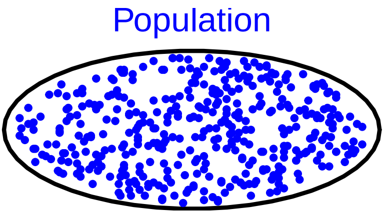

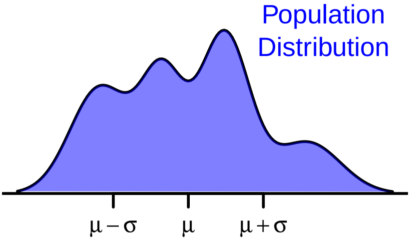

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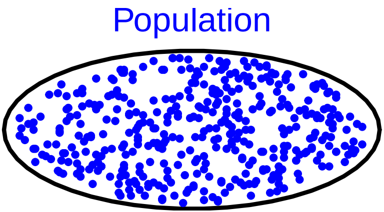

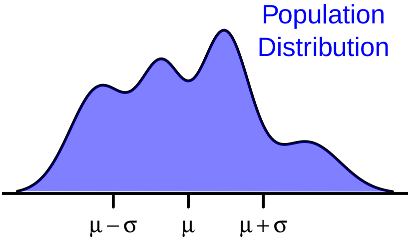

- Suppose we are interested in some numerical characteristic X about individuals in a certain population.
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- The population distribution is arbitrary (not necessarily normal), with a *population mean* μ and a *population SD* σ.
- The goal is to estimate the population mean μ

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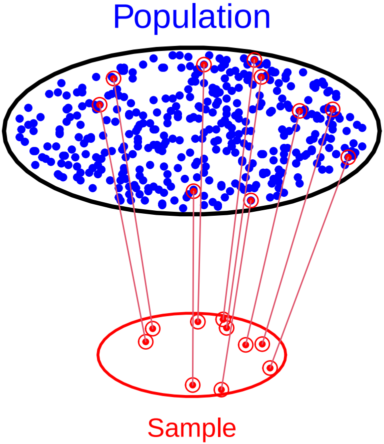
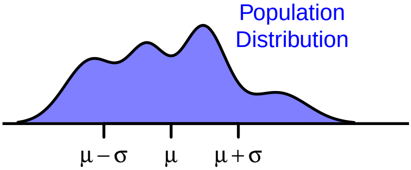
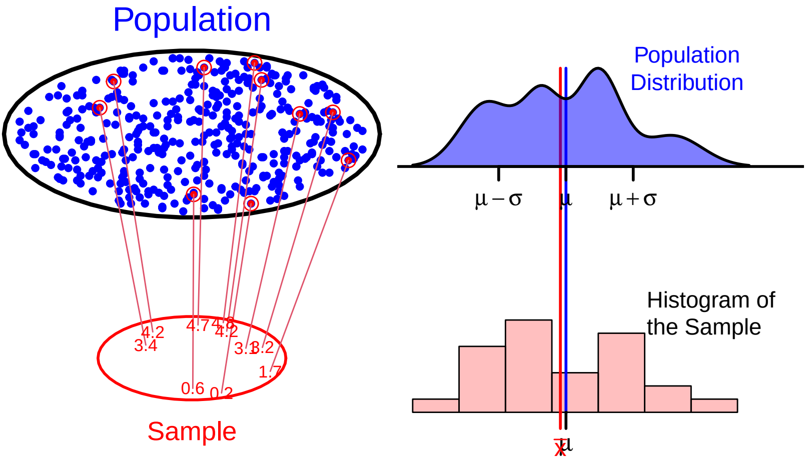




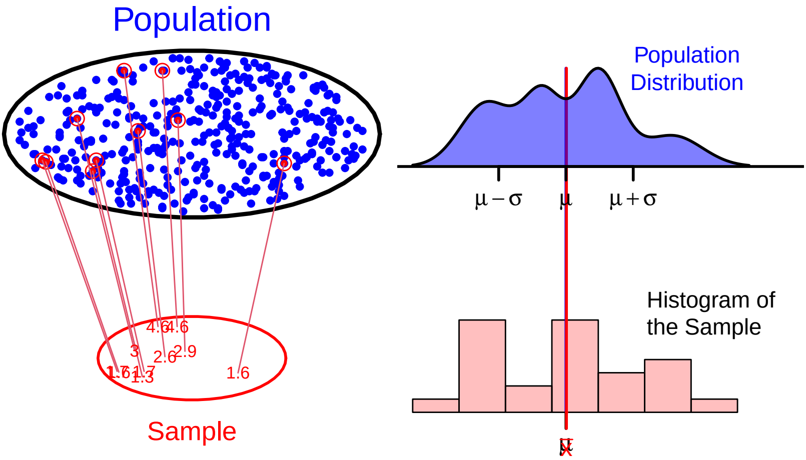
A (simple) random sample is taken from the population.

Variability in Estimation (Review)



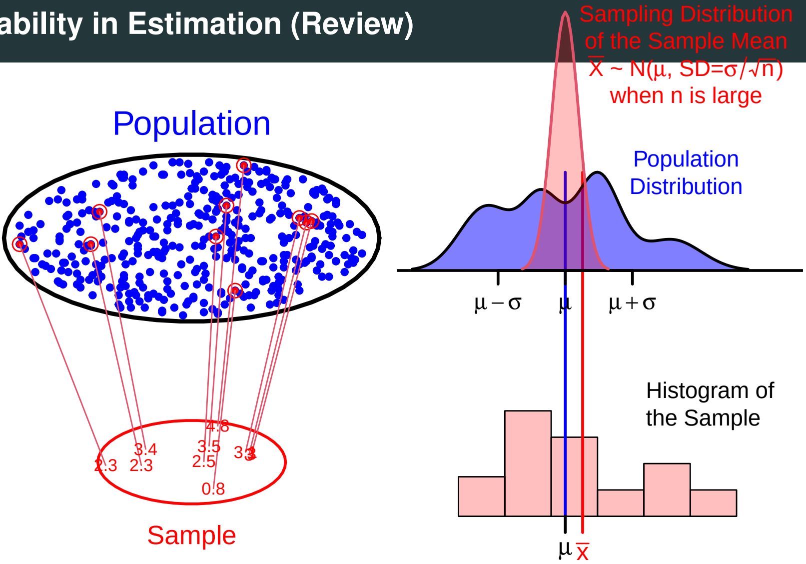
The X-value for each individual in the sample is recorded. One can make a histogram for the recorded X-values.

Variability in Estimation (Review)



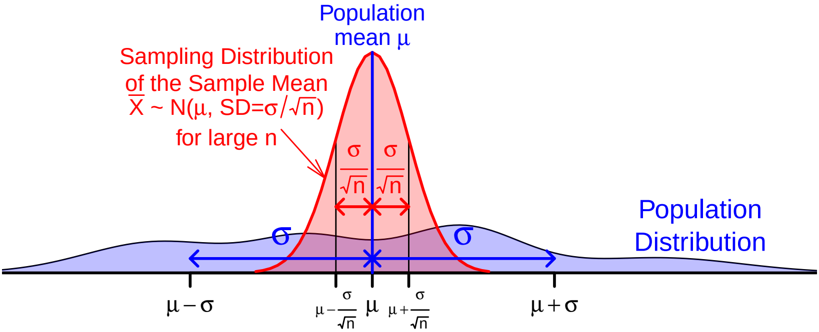
The population mean μ is estimated by the sample mean $\overline{X}$, which will change from sample to sample.

Variability in Estimation (Review)



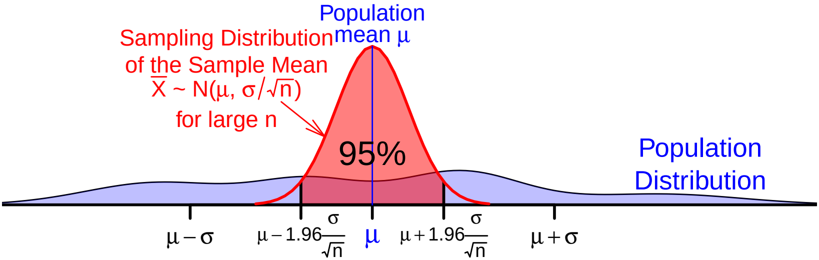
The distribution of the sample mean $\overline{X}$ is approx. normal w/ mean μ and SD $= \sigma/\sqrt{n}$ when n is large by CLT.

σ v.s. $\sigma/\sqrt{n}$



- σ is the SD of the population
- $\frac{\sigma}{\sqrt{n}}$ is the SD of the sampling distribution of $\overline{X}$
    - $\frac{\sigma}{\sqrt{n}}$ is usually called the *standard error (SE)*, to differentiate it from the population SD σ

As a normal random variable will fall within 1.96 SDs from the center 95% of the time, $\overline{X}$ **will fall within** $1.96\frac{\sigma}{\sqrt{n}}$ **from** μ **95% of the time** since $\overline{X}$ is approx. $N(\mu, \sigma/\sqrt{n})$ for large n by CLT.



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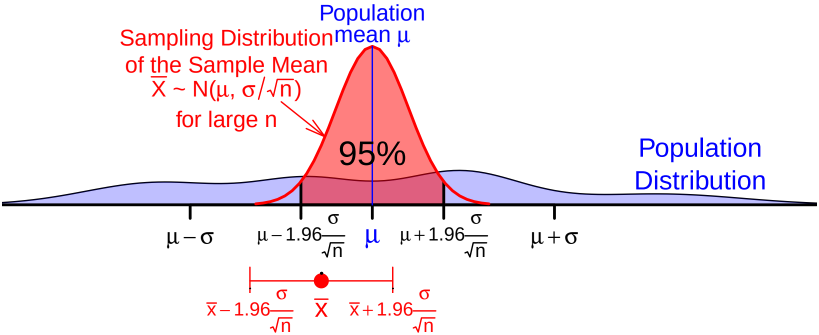

Or equivalently, μ **will be within** $1.96\frac{\sigma}{\sqrt{n}}$ **from** $\overline{X}$ **95% of the time**.

A **95% confidence interval for** μ is hence defined to be

$$\overline{X} \pm 1.96\frac{\sigma}{\sqrt{n}} = \left(\overline{X} - 1.96\frac{\sigma}{\sqrt{n}}, \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

Procedures to Construct a 95% Confidence Interval for μ

1. Take a simple random sample (or i.i.d. sample) of some large enough size n and find the sample mean $\overline{X}$.
2. If n is large, the 95% confidence interval for μ is given by

$$\overline{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

But σ is usually unknown ...

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The unknown population SD σ is replaced by our best guess — *the sample SD* s. So an approximate 95% confidence interval for μ is

$$\overline{X} \pm 1.96 \frac{s}{\sqrt{n}}$$

- However, this replacement is hazardous because
  - s is a poor estimate of σ when the sample size n is small and
  - s is very **sensitive to outliers**
- So we require $n \geq 30$ and sample shouldn't have any outlier nor be too skewed ⇒ Need to check histogram of the data
- We will discuss working with samples where $n < 30$ in the next chapter

Example: Average Number of Exclusive Relationships

A random sample of 50 college students were asked how many exclusive relationships they have been in so far. This sample yielded a mean of 3.2 and a standard deviation of 1.74. Estimate the true average number of exclusive relationships using this sample.

$$\overline{X} = 3.2 \qquad s = 1.74$$

The approximate 95% confidence interval is about

$$\begin{aligned} \overline{X} \pm 1.96 \times \text{SE} &= \overline{X} \pm 1.96 \times \frac{s}{\sqrt{n}} \\ &= 3.2 \pm 1.96 \times \frac{1.74}{\sqrt{50}} \\ &\approx 3.2 \pm 0.5 = (2.7, 3.7) \end{aligned}$$

True or False

True or False and explain: We are 95% confident that the average number of exclusive relationships college students in this sample have been in is between 2.7 and 3.7.

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False. The confidence interval $\overline{X} \pm$ 1.96 SE definitely (100%) contains the *sample mean* $\overline{X}$, not just with probability 95%.

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True or False and explain: 95% of college students have been in 2.7 to 3.7 exclusive relationships.

False. The confidence interval is for covering the population mean μ, not for covering 95% of the entire population. If 95% of college students have been in 2.7 to 3.7 exclusive relationships, the SD won't be as large as 1.74.

True or False

True or False and explain: There is 0.95 probability that the true mean number of exclusive relationships of college students falls in the interval $(2.7, 3.7)$

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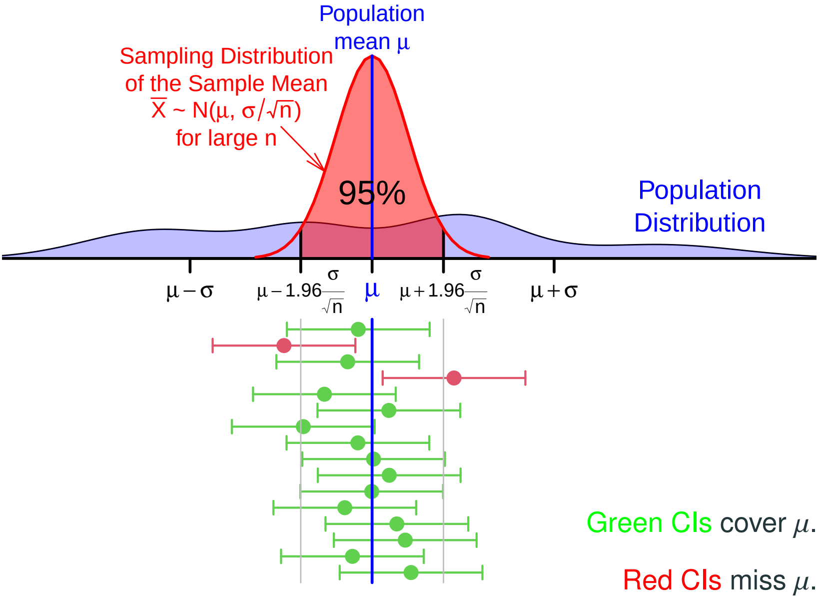
Both are False. The population mean μ is a fixed number, not random. It is either in the interval (2.7, 3.7), or not in the interval. There is no uncertainty involved.

What does “95% confidence” mean?

What is the thing that has a 95% chance to happen?

- It is the **procedure to construct the 95% interval**.
- About 95% of the intervals constructed following the procedure (taking a SRS and then calculating $\overline{X} \pm 1.96\, s/\sqrt{n}$) will cover the true population mean μ.
- After taking the sample and an interval is constructed, the constructed interval either covers μ or it doesn’t. We don’t know. Only God knows.
- Just like lottery, before you pick the numbers and buy a lottery ticket, you have some chance to win the prize. After you get the ticket, you either win or lose.


Population
mean μ
Sampling Distribution
of the Sample Mean
$\overline{X} \sim N(\mu, \sigma/\sqrt{n})$
for large n
95%
Population
Distribution
$\mu - \sigma$
$\mu - 1.96\frac{\sigma}{\sqrt{n}}$
μ
$\mu + 1.96\frac{\sigma}{\sqrt{n}}$
$\mu + \sigma$
Green CIs cover μ.
Red CIs miss μ.

True or False

True or False and explain: If a new random sample of size 50 is taken, we are 95% confident that the new sample mean will be between 2.7 and 3.7.

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False. The confidence interval is for covering the population mean μ, not for covering the mean of another sample. The SE $\sigma/\sqrt{n}$ or $s/\sqrt{n}$ is a typical distance between the sample mean and population mean, not a typical distance between two sample means.

True or False

True or False and explain: This confidence interval $\overline{X} \pm 1.96\, s/\sqrt{n}$ is not valid since the number of exclusive relationships is integer-valued. Neither the population nor sample is normally distributed.

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False. The construction of the CI $\overline{X} \pm 1.96\, s/\sqrt{n}$ only uses the normality of the sampling distribution of the sample mean. Neither the population nor the sample is required to be normally distributed. By the central limit theorem, with a large enough sample size we can assume that the sampling distribution is nearly normal and calculate a confidence interval.

Confidence Intervals at Other Confidence Levels

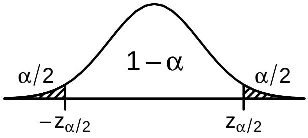
A confidence interval for a population mean μ at confidence level $(1 - \alpha)$ is

$$\text{sample mean} \pm z_{\alpha/2}\text{SE}$$

where $z_{\alpha/2}$ is a number such that

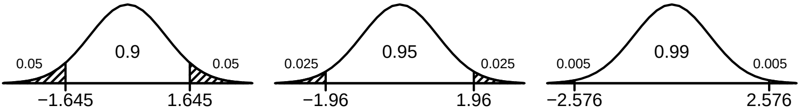
$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha \quad \text{or}$$

where $Z \sim N(0, 1)$.



Commonly used confidence levels:

90% CI, $\alpha = 0.1$	95% CI, $\alpha = 0.05$	99% CI, $\alpha = 0.01$
$z_{0.1/2} \approx 1.645$	$z_{0.05/2} \approx 1.960$	$z_{0.01/2} \approx 2.576$



Example

For the “number of exclusive relationships” example, recall

$$\overline{X} = 3.2, \quad s = 1.74, \quad \text{SE} = \frac{s}{\sqrt{n}} = \frac{1.74}{\sqrt{50}} \approx 0.246$$

- 90% CI: $\overline{X} \pm \mathit{1.645} \times \text{SE} = 3.2 \pm 1.645 \times 0.246 \approx 3.2 \pm 0.40$
- 95% CI: $\overline{X} \pm \mathit{1.96} \times \text{SE} = 3.2 \pm 1.96 \times 0.246 \approx 3.2 \pm 0.48$
- 99% CI: $\overline{X} \pm \mathit{2.576} \times \text{SE} = 3.2 \pm 2.576 \times 0.246 \approx 3.2 \pm 0.63$

How to Choose the Confidence Level?

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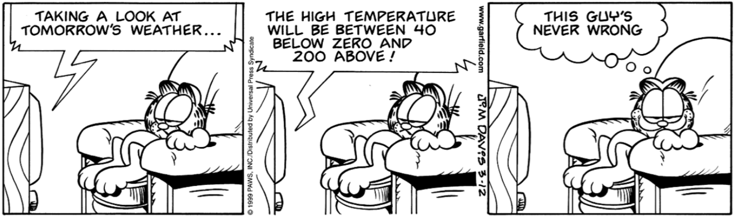

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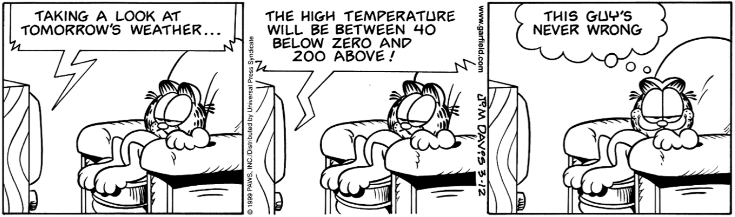

A wide interval may not be informative.

Image source: http://web.as.uky.edu/statistics/users/earo227/misc/garfield_weather.gif