STAT 234 Lecture 15A
Standard Deviation & Sample Variance
(Section 1.4)

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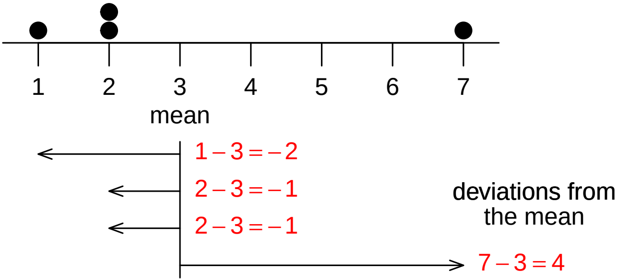
Standard Deviation (SD) — Another Measure of Variability

To understand how *standard deviation (SD)* works, let's use a small data set $\{1, 2, 2, 7\}$ as an example.

- Each of these numbers deviates from the mean $\frac{1+2+2+7}{4} = 3$ by some amount:



Standard Deviation (Cont'd)

- How should we measure the overall size of these deviations?
- Taking their mean doesn't tell us anything about their magnitude
  - since $\sum_i (x_i - \bar{x}) = 0$
- One sensible way is take the average of their absolute values:

$$\frac{|-2| + |-1| + |-1| + |4|}{4} = 2$$

  This is called the mean absolute deviation (MAD), not the SD.
- But for a variety of reasons, statisticians prefer using the root-mean-square as a measure of overall size:

$$\sqrt{\frac{(-2)^2 + (-1)^2 + (-1)^2 + 4^2}{4}} \approx 2.35$$

  but this is still not the (sample) SD.

The formula for the (sample) *standard deviation (SD)* is

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}$$

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- The standard deviation of $\{1, 2, 2, 7\}$ is

$$\sqrt{\frac{(-2)^2 + (-1)^2 + (-1)^2 + 4^2}{4-1}} \approx 2.71$$

(recall we get 2.35 when dividing by $n = 4$)

(Sample) Variance

The square of the (sample) standard deviation is called the *(sample) variance*, denoted as

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

which is roughly the average squared deviation from the mean.

- Note the **sample variance** for a variable in a data set is not the same as the **variance** for a random variable defined to be

$$\mathrm{Var}(X) = \mathrm{E}(X-\mu)^2 = \begin{cases} \sum_x (x-\mu)^2 p(x) & \text{if } X \text{ is discrete} \\ \int (x-\mu)^2 f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Shortcut Formula for Sample Variance

Just like the shortcut formula for the variance of a random variable,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2, \quad \text{where } \mu = \mathrm{E}(X),$$

the sample variance also has its shortcut formula

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1} = \frac{\left(\sum_{i=1}^{n} x_i^2\right) - n(\overline{x})^2}{n-1}.$$

Proof of Shortcut Formula for Sample Variance

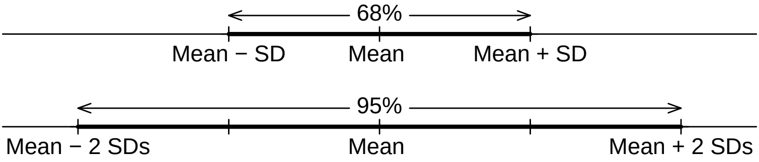
$$
\begin{aligned}
(n-1)s^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2 &= \sum_{i=1}^{n}(x_i^2 - 2\overline{x}x_i + \overline{x}^2) \\
&= \left(\sum_{i=1}^{n} x_i^2\right) - 2\left(\sum_{i=1}^{n} x_i\overline{x}\right) + \underbrace{\sum_{i=1}^{n} \overline{x}^2}_{=n\overline{x}^2} \\
&= \left(\sum_{i=1}^{n} x_i^2\right) - 2\overline{x}\underbrace{\left(\sum_{i=1}^{n} x_i\right)}_{=n\overline{x}} + n\overline{x}^2 \\
&= \left(\sum_{i=1}^{n} x_i^2\right) - 2n\overline{x}^2 + n\overline{x}^2 \\
&= \left(\sum_{i=1}^{n} x_i^2\right) - n\overline{x}^2
\end{aligned}
$$

Now we have proved the shortcut formula for sample variance

$$
s^2 = \frac{\left(\sum_{i=1}^{n} x_i^2\right) - n\overline{x}^2}{n-1}.
$$

The 68% and 95% Rule of SD

- Roughly 68% of the observations will be within 1 SD away from the mean
- Roughly 95% will be with 2 SD away from the mean



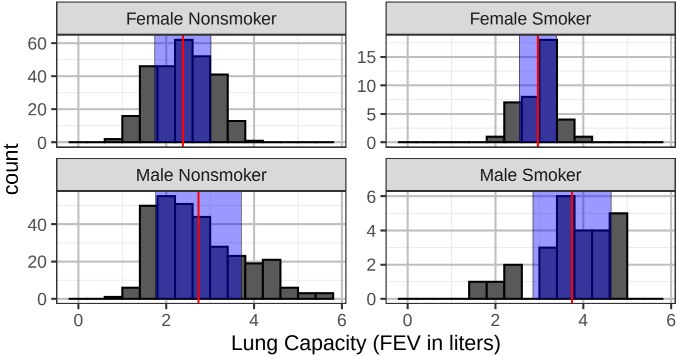
The value 68% and 95% comes from the standard normal distribution that

$$P(-1 < Z < 1) \approx 0.68 \quad \text{and} \quad P(-2 < Z < 2) \approx 0.95 \quad \text{if } Z \sim N(0, 1).$$

The 68% and 95% rule work very well for bell-shaped data, and reasonably well for unimodal and not seriously skewed data, but not for all data.

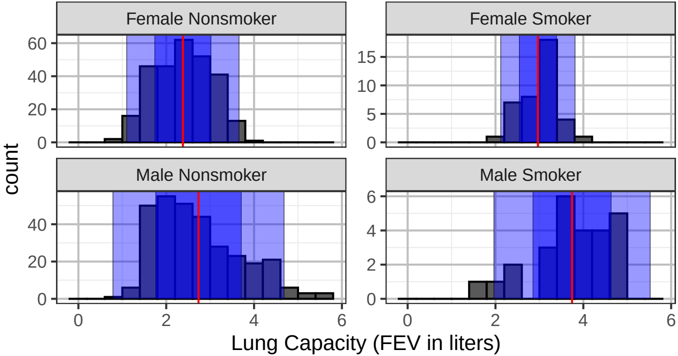
68% Rule for the FEV Data

Sex/Smoke	Mean	SD	Count	proportion within *1 SD* from mean
Female Nonsmoker	2.379	0.639	279	$170/279 \approx 60.9\%$
Female Smoker	2.966	0.423	39	$27/39 \approx 69.2\%$
Male Nonsmoker	2.734	0.974	310	$203/310 \approx 65.5\%$
Male Smoker	3.743	0.889	26	$17/26 \approx 65.4\%$



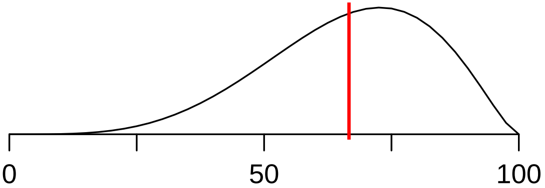
95% Rule for the FEV Data

Sex/Smoke	Mean	SD	Count	proportion within *2 SDs* from mean
Female Nonsmoker	2.379	0.639	279	$270/279 \approx 96.8\%$
Female Smoker	2.966	0.423	39	$38/39 \approx 97.4\%$
Male Nonsmoker	2.734	0.974	310	$298/310 \approx 96.1\%$
Male Smoker	3.743	0.889	26	$24/26 \approx 92.3\%$



Exercise

Is the SD of the histogram below closest to 5, 15, or 50?



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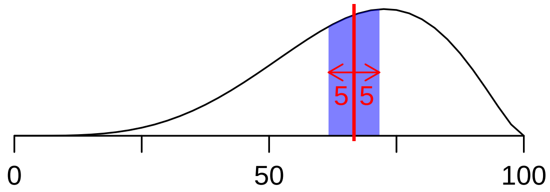

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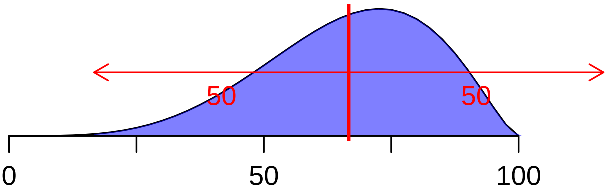

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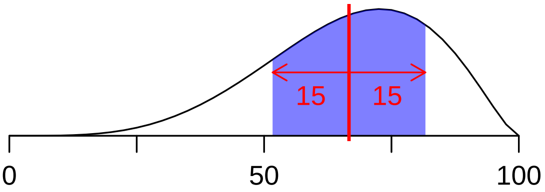

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- and what if SD < 0?
- SD is very sensitive to outliers.
- SD has the same units of measurement as the original observations, while the variances in the square of these units.

Linear Transformation and Sample SD/Variance

Sample SD/variance has the same scaling properties as the SD/variance of random variables.

If $y_i = ax_i + b$ for all $i = 1, 2, \ldots, n$, then

- $s_y^2 = a^2 s_x^2$
- $s_y = |a| s_x$

Recap: Graphical Summaries Numerical Variables

Histograms

- Shape (skewness, modality), outlier, center, spread

Box-plots

- Graphical display of the five-number summary + 1.5 IQR rule
- can reveal skewness but not modality

Recap: Numerical Summaries of Numerical Variables

Common measure of **center**:

- Mean
- Median

Common measure **spread**:

- Range: max – min
- SD
- IQR = $Q_3 - Q_1$

Center and **spread** are important summaries of a distribution. But they don't tell us about the modality and skewness of a distribution, and whether there are outliers.

Always check the histogram!