STAT 234 Lecture 10A
Independent Random Variables
Section 5.1

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Independent Random Variables (L09 Review)

- Recall that two events A and B are *independent* if

$$P(A \cap B) = P(A)P(B)$$

- Two random variables X and Y are *independent* if

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

  for any sets A and B.

- It can be show that two random variables X and Y are *independent* if and only if

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{if } X \text{ and } Y \text{ are discrete}$$

$$f(x, y) = f_X(x)f_Y(y) \qquad \text{if } X \text{ and } Y \text{ are continuous}$$

  for all x and y, i.e., the joint distribution of X and Y is the product of their marginal distribution.

Are X and Y Independent? (L09 Review)

	$p(x,y)$	y = 1	2	3	
x	1	0.05	0.10	0.05	
	2	0.10	0.40	0.10	
	3	0.05	0.10	0.05	

Are X and Y Independent? (L09 Review)

	$p(x,y)$	y = 1	2	3	$p_X(x)$
	1	0.05	0.10	0.05	0.20
x	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

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1. Find the marginal distributions
2. Check whether

$$p(x,y) = p_X(x)p_Y(y)$$

   for all possible x, y pairs.

   - $p(1,1) = 0.05 \neq 0.2 \times 0.2 = p_X(1)p_Y(1)$.
   - X and Y are NOT independent.

Finding Joint pmf From Marginal pmf's When Independent

Given the marginal pmfs of two **independent** r.v.'s, X and Y, find their joint pmf.

	$p(x,y)$	y = 1	2	3	$p_X(x)$
	1				0.2
x	2				0.6
	3				0.2
	$p_Y(y)$	0.2	0.6	0.2	

Since X and Y are **independent**,

1. $p(1,1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
2. also $p(1,2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12$.
3. Repeat filling the blank for $p(x,y)$ by $p_X(x)p_Y(y)$ for all x, y pairs.

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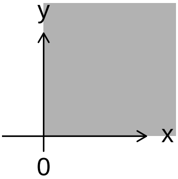
Finding Joint pdf From Marginal pdf's When Independent

Suppose the lifetimes X and Y of Batteries A and B are independent with pdfs

$$f_X(x) = e^{-x} \quad \text{and} \quad f_Y(y) = 2e^{-2y},$$

for $0 < x, y < \infty$, then their joint pdf is

$$f(x, y) = f_X(x) f_Y(y) = 2e^{-(x+2y)}, \quad 0 < x, y < \infty.$$



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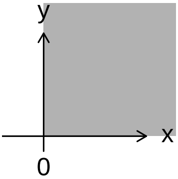

Question: $P(X < Y) = P(\text{Battery A dies before Battery B}) = ?$

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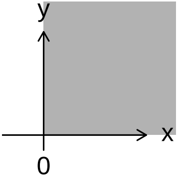
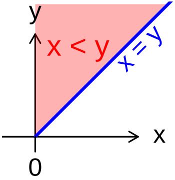


<u>Question</u>: $P(X < Y) = P(\text{Battery A dies before Battery B}) = ?$

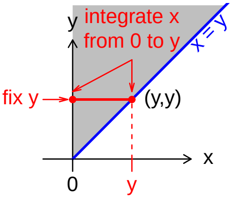
$$P(X < Y) = \iint_{x<y} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{0<x<y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y$$



Method 1

$$\iint\limits_{0<x<y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y = \int_0^\infty \int_0^y 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y$$

where $\int_0^y 2e^{-(x+2y)}\mathrm{d}x = -2e^{-(x+2y)}\Big|_{x=0}^{x=y} = 2(e^{-2y} - e^{-3y})$.



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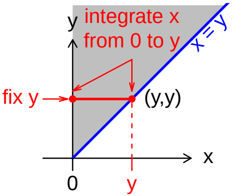

Putting it back to the double integral, we get

$$\int_0^\infty \int_0^y 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y = \int_0^\infty 2(e^{-2y} - e^{-3y})\mathrm{d}y = -e^{-2y} - \frac{2}{3}e^{-3y}\Big|_0^\infty = \frac{1}{3}$$

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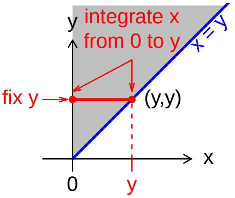

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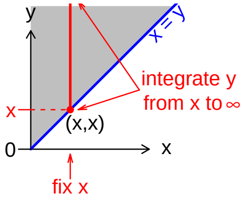
Method 2

$$\iint_{0<x<y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y = \int_0^\infty \int_x^\infty 2e^{-(x+2y)}\mathrm{d}y\mathrm{d}x$$

where $\int_x^\infty 2e^{-(x+2y)}\mathrm{d}y = -e^{-(x+2y)}\Big|_{y=x}^{y=\infty} = e^{-3x}$.



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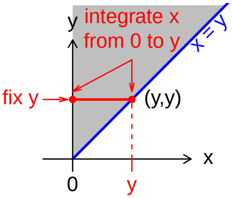

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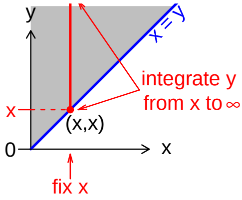

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Example - Are X & Y Independent?

Suppose the joint pdf of X, Y is

$$f(x, y) = 6xy^2, \qquad \text{for } 0 \leq x, y \leq 1.$$

The marginal pdf of X is

$$f_X(x) = \int_0^1 6xy^2 \, \mathrm{d}y = 2xy^3 \Big|_{y=0}^{y=1} = 2x(1^3 - 0^3) = 2x, \quad 0 < x < 1.$$

The marginal pdf of Y is

$$f_Y(y) = \int_0^1 6xy^2 \, \mathrm{d}x = 3x^2y^2 \Big|_{x=0}^{1} = 3y^2(1^2 - 0^2) = 3y^2, \quad 0 < y < 1.$$

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The marginal pdf of Y is

$$f_Y(y) = \int_0^1 6xy^2 \, \mathrm{d}x = 3x^2y^2 \Big|_{x=0}^{1} = 3y^2(1^2 - 0^2) = 3y^2, \quad 0 < y < 1.$$

Are X and Y independent?

- Yes, since $f(x, y) = 6xy^2 = (2x)(3y^2) = f_X(x)f_Y(y)$ for all $0 \leq x, y \leq 1$ and $f(x, y) = 0 = f_X(x)f_Y(y)$ elsewhere.

A Simple Criterion for Checking Independence

So far, it seems like one must find the marginal distributions before checking independence. However, there is an easier way...

A Simple Criterion: X and Y are independent if the joint pmf/pdf can be written as the product of a function of x and a function of y.

$$f(x, y) = g(x)h(y), \quad \text{for all } x, y.$$

Here $g(x) \geq 0$ and $h(y) \geq 0$ are **not necessarily pmfs/pdfs**.

Are They Independent?

1. $p(x, y) = \dfrac{x + y}{36}$ for $x, y \in \{1, 2, 3\}$.
    - can't be factored, X and Y are NOT independent.

2. $p(x, y) = e^{-2}/(x!y!)$, for $x, y \in \{0, 1, 2, \ldots\}$.
    - factors into $g(x) = e^{-1}/x!$ and $h(y) = e^{-1}/y!$, so independent.

3. $f(x, y) = 8xy$ for $0 \leq x < y \leq 1$.
    - Does it factors into $g(x) = 8x$ and $h(y) = y$?

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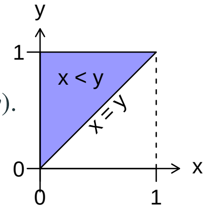
3. $f(x, y) = 8xy$ for $0 \le x < y \le 1$.
    - Does it factors into $g(x) = 8x$ and $h(y) = y$?
    - Watch out! When $x > y$, $f(x, y) = 0 \neq g(x)h(y)$.
    - X and Y are NOT independent.



Proof of the Simple Criterion (okay to skip)

We prove the discrete case. The continuous case is similar. The marginal pdf of Y is

$$\begin{aligned} p_Y(y) = \sum\nolimits_x p(x,y) &= \sum\nolimits_x g(x)h(y) \\ &= h(y)\sum\nolimits_x g(x) = c_1 h(y), \end{aligned}$$

in which c_1 is the constant $\sum_x g(x)$. Similarly, one can show $p_X(x) = c_2 g(x)$ where $c_2 = \sum_y h(y)$. Note that

$$\begin{aligned} c_1 c_2 = \sum\nolimits_x g(x) \sum\nolimits_y h(y) &= \sum\nolimits_x \sum\nolimits_y g(x)h(y) \\ &= \sum\nolimits_x \sum\nolimits_y f(x,y) = 1 \end{aligned}$$

since $p(x,y)$ is a joint pmf.

Thus $p_X(x)p_Y(x) = c_1 c_2 g(x)h(y) = g(x)h(y) = f(x,y)$.

Independence of Several Random Variables

- More generally, a sequence of random variables $X_1, X_2, \ldots, X_n$ are **(mutually) independent** if and only if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n).$$

  for all sequence of events $A_1, A_2, \ldots$.
- Equivalently, the random variables $X_1, X_2, \ldots, X_n$ are **(mutually) independent** if and only if their joint distributions factors into the product of their marginal distributions.

$$p(x_1, x_2, \ldots, x_n) = p_1(x_1)p_2(x_2)\ldots p_n(x_n) \qquad \text{for discrete rv's}$$
$$f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_2)\ldots f_n(x_n) \qquad \text{for continuous rv's}$$

  for all $x_1, x_2, \ldots, x_n$.