STAT 234 Lecture 9
Joint Distributions of Random Variables
Section 5.1

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Why Consider Two or More Random Variables?

- Our focus so far has been on the distribution of a single random variable.
- In many situations, there are two or more variables of interest, and we want to know how they are related. For example, I am interested to know
  - X_1: the number of hours spent on studying per week
  - X_2: final grade of stat234.
- Since the *relationship* is important, we cannot study them separately and need to consider them *jointly*.

Joint Probability Distribution for Discrete R.V.

Joint Distribution of Two Discrete Random Variables

The *joint probability mass function (joint pmf)*, or, simply the *joint distribution*, of two discrete r.v. X and Y is defined as

$$p(x, y) = P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\}).$$

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Properties of the joint probability distribution:

1. $p(x, y) \geq 0$.
2. Define the probability for an event A as,

$$P(A) = P((x, y) \in A) = \sum_{(x,y) \in A} p(x, y)\,.$$

3. If we set $A = S$ in (2), then

$$P(S) = \sum_{x} \sum_{y} p(x, y) = 1\,.$$

Exercise 1 — Gas Station (p.242 in MMSA)

A gas station has both **self-service** and **full-service** islands, each with a single regular unleaded pump with 2 hoses.

X = the # of hoses in use on the self-service island, and

Y = the # of hoses in use on the full-service island

The joint pmf of X and Y:

		Y (full-service)		
	$p(x, y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $P(X = 2 \text{ and } Y = 1)$?

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What is $P(X = 2$ and $Y = 1)$? $p(2, 1) = 0.14$

Exercise 1 — Gas Station (2)

	$p(x, y)$	Y (full-service) 0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $P(X \leq 1$ and $Y \leq 1)$?

Exercise 1 — Gas Station (2)

	$p(x,y)$	Y (full-service) 0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $P(X \leq 1 \text{ and } Y \leq 1)$?

$$P(X=0,Y=0)+P(X=0,Y=1)+P(X=1,Y=0)+P(X=1,Y=1)$$

$$= p(0,0)+p(0,1)+p(1,0)+p(1,1)$$

$$= 0.10+0.04+0.08+0.20 = 0.42$$

Exercise 1 — Gas Station (3)

		Y (full-service)		
	$p(x, y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is the probability that more self-service hoses in use than full service hoses $P(X > Y)$?

Exercise 1 — Gas Station (3)

		Y (full-service)		
	$p(x,y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is the probability that more self-service hoses in use than full service hoses $P(X > Y)$?

$$P(X = 1, Y = 0) + P(X = 2, Y = 1) + P(X = 2, Y = 0)$$

$$= p(1, 0) + p(2, 1) + p(2, 1)$$

$$= 0.08 + 0.06 + 0.14 = 0.28$$

Marginal Distribution

Obtaining pmf of X From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2	Row Sum
X	0	0.10	0.04	0.02	
	1	0.08	0.20	0.06	
	2	0.06	0.14	0.30	

$P(X = 0) =$

Obtaining pmf of X From the Joint Distribution of (X, Y)

	$p(x, y)$	Y = 0	1	2	Row Sum
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	
	2	0.06	0.14	0.30	

$$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$$
$$= 0.10 + 0.04 + 0.02 = 0.16$$

Obtaining pmf of X From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2	Row Sum
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
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Likewise,

$$P(X=1) = 0.08 + 0.20 + 0.06 = 0.34$$

Obtaining pmf of X From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2	Row Sum
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

$$
\begin{aligned}
P(X=0) &= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) \\
&= 0.10 + 0.04 + 0.02 = 0.16
\end{aligned}
$$

Likewise,

$$P(X=1) = 0.08 + 0.20 + 0.06 = 0.34$$

$$P(X=2) = 0.06 + 0.14 + 0.30 = 0.50$$

Obtaining pmf of X From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2	Row Sum $p_X(x)$
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

$$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$$
$$= 0.10 + 0.04 + 0.02 = 0.16$$

Likewise,

$$P(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$$
$$P(X = 2) = 0.06 + 0.14 + 0.30 = 0.50$$

The pmf $p_X(x)$ of X is thus

x	0	1	2
$p_X(x)$	0.16	0.34	0.50

Obtaining pmf of Y From the Joint Distribution of (X, Y)

	$p(x, y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum				

$P(Y = 0) =$

Obtaining pmf of Y From the Joint Distribution of (X, Y)

		Y		
	$p(x, y)$	0	1	2
	0	0.10	0.04	0.02
X	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum		0.24		

$$P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)$$

$$= 0.10 + 0.08 + 0.06 = 0.24$$

Obtaining pmf of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30
Column sum		0.24	0.38	

$$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=2, Y=0)$$
$$= 0.10 + 0.08 + 0.06 = 0.24$$

Likewise,

$$P(Y=1) = 0.04 + 0.20 + 0.14 = 0.38$$

Obtaining pmf of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
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Column sum		0.24	0.38	0.38

$$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=2, Y=0)$$
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Likewise,

$$P(Y=1) = 0.04 + 0.20 + 0.14 = 0.38$$
$$P(Y=2) = 0.02 + 0.06 + 0.30 = 0.38$$

Obtaining pmf of Y From the Joint Distribution of (X, Y)

	$p(x,y)$	Y = 0	1	2
X	0	0.10	0.04	0.02
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Column sum	$p_Y(y)$	0.24	0.38	0.38

$$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=2, Y=0)$$
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Likewise,

$$P(Y=1) = 0.04 + 0.20 + 0.14 = 0.38$$
$$P(Y=2) = 0.02 + 0.06 + 0.30 = 0.38$$

The pmf $p_Y(y)$ of Y is thus

y	0	1	2
$p_Y(y)$	0.24	0.38	0.38

Marginal Distribution

The **marginal probability mass functions (marginal pmf's)** of X and of Y are obtained by summing $p(x,y)$ over values of *the other variable*.

$$p_X(x) = \sum_y p(x,y), \quad p_Y(y) = \sum_x p(x,y).$$

Example: Gas Station

	$p(x,y)$	Y = 0	1	2	Row Sum $p_X(x)$
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
Column sum	$p_Y(y)$	0.24	0.38	0.38	

We call them **marginal distributions** because they show up at the table margins when the joint distribution is written in a tabular form

Joint Distribution of Continuous Random Variables

Joint Distribution of Two Continuous Random Variables

Let X and Y be continuous rv. Then $f(x, y)$ is their *joint probability density function* or *joint pdf* for X and Y if for any two-dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

In particular, if A is the two-dimensional rectangle $\{a \leq x \leq b, c \leq y \leq d\}$, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

Conditions for a joint pdf

- It must be nonnegative: $f(x, y) \geq 0$ for all x and y
- $\iint f(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$

Example 5.5 on p.237-238 of MMSA

- Each can of mixed nuts contains *almonds*, *cashews*, and *peanuts*

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- In a randomly selected can, let

  $X =$ the weight of almonds, and $Y =$ the weight of cashews.

  The weight of peanuts in the can is thus $(1 - X - Y)$

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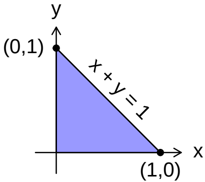
- In a randomly selected can, let

  $X =$ the weight of almonds, and $Y =$ the weight of cashews.

  The weight of peanuts in the can is thus $(1 - X - Y)$
- Natural constraints on X & Y:

$$0 \le X \le 1, 0 \le Y \le 1, X + Y < 1$$



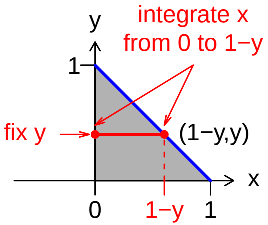
- Joint pdf of X & Y:

$$f(x,y) = \begin{cases} 24xy & \text{if } 0 \le x \le 1, 0 \le y \le 1, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Checking Conditions on a Joint PDF

Clearly, $f(x,y) \geq 0$. It remains to check $\iint f(x,y)\mathrm{d}x\mathrm{d}y = 1$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y$$



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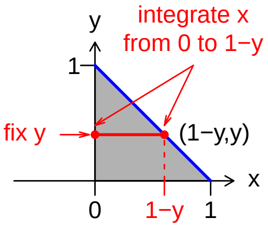

To compute the double integral above,

1. hold one variable fixed (e.g., y)

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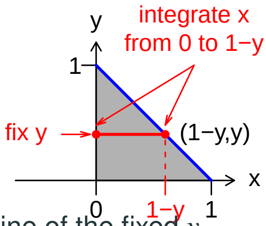


To compute the double integral above,

1. hold one variable fixed (e.g., y)
2. integrate the other variable x along the line of the fixed y
   - key: express the end points of the line in terms of the fixed y, which will be the upper and lower limits for the integral over x

$$\int_0^{1-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=1-y} = 12(1-y)^2y$$

3. integrate the variable y that is fixed in the prior steps

$$\int_0^1\int_0^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 12(1-y)^2y\mathrm{d}y = 6y^2 - 8y^3 + 3y^4\Big|_0^1 = 1.$$

Finding Probabilities From the Joint PDF $P(X > 0.3)$

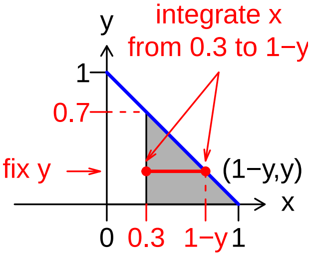
What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

Finding Probabilities From the Joint PDF $P(X > 0.3)$

What is $P(X > 0.3) = P$(at least 30% almonds in a can)?

$$P(X > 0.3) = \iint_{x>0.3} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7}\int_{0.3}^{1-y} 24xy\,\mathrm{d}x\mathrm{d}y$$

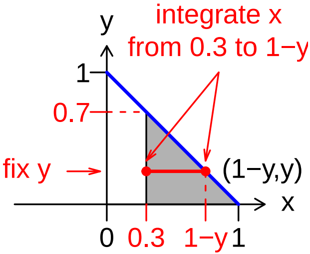


Finding Probabilities From the Joint PDF $P(X > 0.3)$

What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

$$P(X > 0.3) = \iint_{x>0.3} f(x, y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7} \int_{0.3}^{1-y} 24xy \, \mathrm{d}x\mathrm{d}y$$



where

$$\int_{0.3}^{1-y} 24xy \, \mathrm{d}x = 12x^2y\Big|_{x=0.3}^{x=1-y}$$

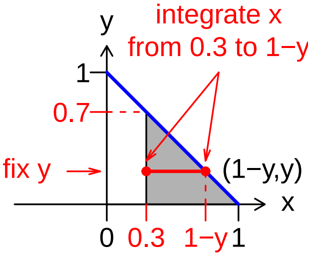
$$= 12((1 - y)^2 - 0.3^2)y = 12(0.91y - 2y^2 + y^3).$$

Finding Probabilities From the Joint PDF $P(X > 0.3)$

What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

$$P(X > 0.3) = \iint_{x>0.3} f(x, y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7} \int_{0.3}^{1-y} 24xy\,\mathrm{d}x\mathrm{d}y$$



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$$= 12((1-y)^2 - 0.3^2)y = 12(0.91y - 2y^2 + y^3).$$

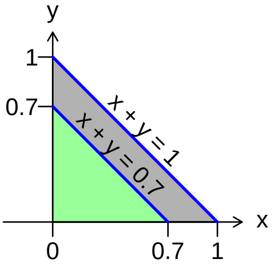
Putting it back to the double integral, we get

$$\int_0^{0.7}\int_{0.3}^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^{0.7} 12(0.91y - 2y^2 + y^3)\mathrm{d}y$$

$$= 5.46y^2 - 8y^3 + 3y^4\Big|_0^{0.7} = 0.6517.$$

Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?



$P(\text{less than 30\% are Peanuts})$

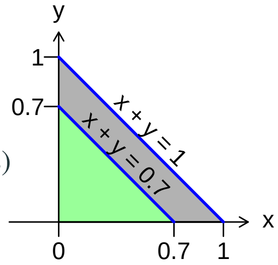
$=$

$=$

$=$

Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

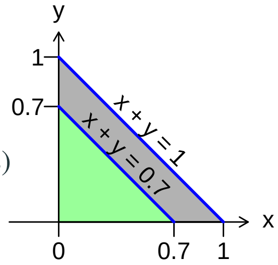
What is the probability that less than 30% are peanuts in a randomly selected can?



$$
\begin{aligned}
&P(\text{less than 30\% are Peanuts}) \\
&= P(\text{at least 70\% are almonds or cashews}) \\
&= \\
&=
\end{aligned}
$$

Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

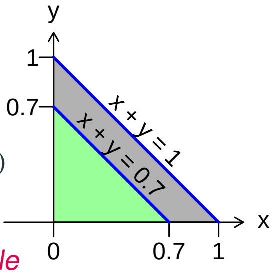
What is the probability that less than 30% are peanuts in a randomly selected can?



$$\begin{aligned}
&P(\text{less than 30\% are Peanuts}) \\
&= P(\text{at least 70\% are almonds or cashews}) \\
&= P(X + Y > 0.7) \\
&=
\end{aligned}$$

Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

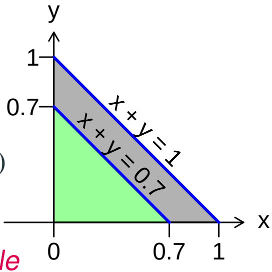
What is the probability that less than 30% are peanuts in a randomly selected can?



$$
\begin{aligned}
& P(\text{less than 30\% are Peanuts}) \\
&= P(\text{at least 70\% are almonds or cashews}) \\
&= P(X + Y > 0.7) \\
&= 1 - P(X + Y \leq 0.7) \quad \text{by } \textit{Complement Rule}
\end{aligned}
$$

Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?



$$\begin{aligned} &P(\text{less than 30\% are Peanuts}) \\ = &P(\text{at least 70\% are almonds or cashews}) \\ = &P(X + Y > 0.7) \\ = &1 - P(X + Y \leq 0.7) \quad \text{by } \textit{Complement Rule} \end{aligned}$$

where

$$P(X + Y > 0.7) = \text{integral of } f(x, y) \text{ over the } \textbf{gray} \text{ region}$$

$$P(X + Y < 0.7) = \text{integral of } f(x, y) \text{ over the } \textbf{green} \text{ region}$$

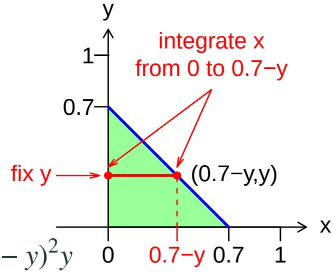
Finding Probabilities From the Joint PDF $P(X+Y>0.7)$ (Cont'd)

$$P(X+Y<0.7) = \iint_{x+y<0.7} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\mathrm{d}y$$

where

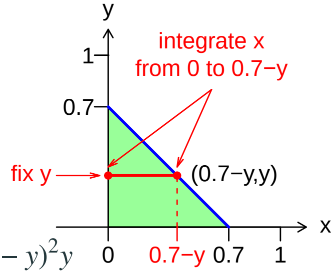
$$\int_0^{0.7-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=0.7-y} = 12(0.7-y)^2y$$



Finding Probabilities From the Joint PDF $P(X+Y>0.7)$ (Cont'd)

$$P(X+Y<0.7) = \iint_{x+y<0.7} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\mathrm{d}y$$



where

$$\int_0^{0.7-y} 24xy\,\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=0.7-y} = 12(0.7-y)^2y$$

Putting it back to the double integral, we get

$$\int_0^{0.7}\int_0^{0.7-y} 24xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^{0.7} 12(0.7-y)^2y\mathrm{d}y = \int_0^{0.7}(-4y)\mathrm{d}(0.7-y)^3$$

$$= -4y(0.7-y)^3\Big|_0^{0.7} + \int_0^{0.7} 4(0.7-y)^3\mathrm{d}y$$

$$= 0 - (0.7-y)^4\Big|_0^{0.7} = (0.7)^4 = 0.2401.$$

Hence, P(less than 30% peanut) $= 1 - 0.2401 = 0.7599$.

Obtaining Marginal PDF's From Joint PDF

Given the joint pdf $f(x, y)$ of two continuous random variables, the *marginal probability density function (p)*, or simply the *marginal density*, of X and Y, can be obtained by *integrating the joint pdf over the other variable*.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y, \quad \text{for } -\infty < x < \infty,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x, \quad \text{for } -\infty < y < \infty.$$

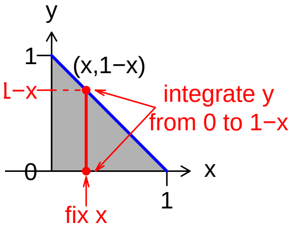
Recall the **marginal pmf's** of discrete random variables are obtained by *summing the joint pmf over values of the other variable*.

$$p_X(x) = \sum_y p(x, y), \quad p_Y(y) = \sum_x p(x, y).$$

Back to Example 5.5

The marginal pdfs of X (almond) is

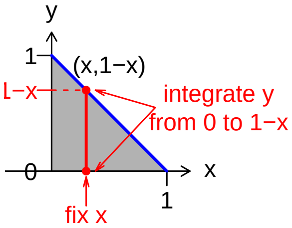
$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y \\
&= \int_0^{1-x} 24xy\,\mathrm{d}y = 12xy^2\Big|_{y=0}^{y=1-x} \\
&= 12x(1-x)^2, \text{ for } 0 \le x \le 1.
\end{aligned}
$$



Back to Example 5.5

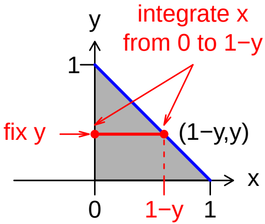
The marginal pdfs of X (almond) is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y \\
&= \int_0^{1-x} 24xy\mathrm{d}y = 12xy^2\Big|_{y=0}^{y=1-x} \\
&= 12x(1-x)^2, \text{ for } 0 \le x \le 1.
\end{aligned}
$$



The marginal pdfs of Y (cashew) is

$$
\begin{aligned}
f_Y(y) &= \int_{\infty}^{\infty} f(x,y)\mathrm{d}x \\
&= \int_0^{1-y} 24xy\mathrm{d}x = 12x^2y\Big|_{x=0}^{x=1-y} \\
&= 12y(1-y)^2, \text{ for } 0 \le y \le 1\,.
\end{aligned}
$$



Independent Random Variables

Independent Random Variables

- Recall that two events A and B are *independent* if

$$P(A \cap B) = P(A)P(B)$$

- Two random variables X and Y are *independent* if

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

  for any sets A and B.
- It can be show that two random variables X and Y are *independent* if and only if

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{if } X \text{ and } Y \text{ are discrete}$$

$$f(x, y) = f_X(x)f_Y(y) \qquad \text{if } X \text{ and } Y \text{ are continuous}$$

  for all x and y, i.e., the joint distribution of X and Y is the product of their marginal distribution.

Are X and Y Independent?

	$f(x, y)$	y = 1	2	3	
	1	0.05	0.10	0.05	
x	2	0.10	0.40	0.10	
	3	0.05	0.10	0.05	

Are X and Y Independent?

	$f(x,y)$	y = 1	2	3	$f_X(x)$
x	1	0.05	0.10	0.05	0.20
	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions

Are X and Y Independent?

		y			
	$f(x,y)$	1	2	3	$f_X(x)$
	1	0.05	0.10	0.05	0.20
x	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions
2. Check whether

$$p(x,y) = p_X(x)p_Y(y)$$

Are X and Y Independent?

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	2	0.10	0.40	0.10	0.60
	3	0.05	0.10	0.05	0.20
	$f_Y(y)$	0.20	0.60	0.20	

1. Find the marginal distributions
2. Check whether

$$p(x,y) = p_X(x)p_Y(y)$$

   for all possible x, y pairs.

   - $p(1,1) = 0.05 \neq 0.2 \times 0.2 = p_X(1)p_Y(1)$.
   - X and Y are NOT independent.

Finding Joint pmf From Marginal pmf's When Independent

Given the marginal pmfs of two **independent** r.v.'s, X and Y, find their joint pmf.

	$p(x,y)$	y = 1	2	3	$f_X(x)$
	1				0.2
x	2				0.6
	3				0.2
	$f_Y(y)$	0.2	0.6	0.2	

Since X and Y are **independent**,

1. $p(1,1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
2. also $p(1,2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12$.
3. Repeat filling the blank for $p(x,y)$ by $p_X(x)p_Y(y)$ for all x, y pairs.

Finding Joint pmf From Marginal pmf's When Independent

Given the marginal pmfs of two **independent** r.v.'s, X and Y, find their joint pmf.

	$p(x,y)$	y = 1	2	3	$f_X(x)$
x	1	0.04			0.2
	2				0.6
	3				0.2
	$f_Y(y)$	0.2	0.6	0.2	

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	2				0.6
	3				0.2
	$f_Y(y)$	0.2	0.6	0.2	

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Finding Joint pmf From Marginal pmf's When Independent

Given the marginal pmfs of two **independent** r.v.'s, X and Y, find their joint pmf.

	$p(x,y)$	y = 1	2	3	$f_X(x)$
	1	0.04	0.12	0.04	0.2
x	2	0.12	0.36	0.12	0.6
	3	0.04	0.12	0.04	0.2
	$f_Y(y)$	0.2	0.6	0.2	

Since X and Y are **independent**,

1. $p(1,1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
2. also $p(1,2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12$.
3. Repeat filling the blank for $p(x,y)$ by $p_X(x)p_Y(y)$ for all x, y pairs.

Finding Joint pdf From Marginal pdf's When Independent

If X and Y are independent with marginal pdfs

$$f_X(x) = e^{-x} \quad \text{and} \quad f_Y(y) = 2e^{-2y},$$

for $0 < x, y < \infty$, then their joint pdf is

$$f(x, y) = f_X(x) f_Y(y) = 2e^{-(x+2y)}, \quad 0 < x, y < \infty.$$