STAT 234 Lecture 6B
Continuous Random Variables
Section 4.1

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Coverage

Continuous Random Variables (Section 4.1 of MMSA)

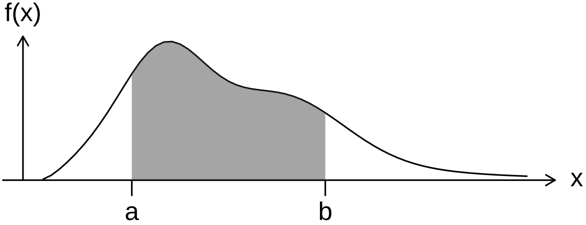
- probability density function (pdf)
- cumulative distribution function (cdf)

Continuous Random Variables

Continuous Random Variables

A random variable X is said to have a *continuous distribution* if there exists a non-negative function f such that

$$P(a < X \leq b) = \int_a^b f(x)\,dx, \quad \text{for all } -\infty \leq a < b \leq \infty.$$



Here f is called the *probability density function* (*pdf*), the *density curve*, or the *density* of X.

Conditions of pdf

A pdf $f(x)$ can be of any imaginable shape but must satisfy the following:

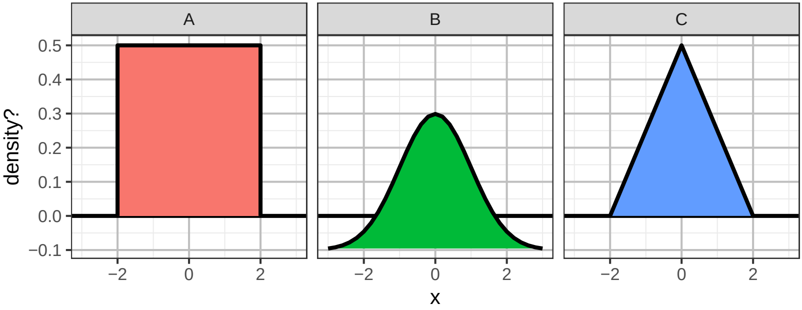
- It must be *nonnegative*

$$f(x) \geq 0 \text{ for all } x$$

- The total area under the pdf must be 1

$$\int_{-\infty}^{\infty} f(x)\,dx = P(-\infty < X \leq \infty) = 1$$

For each of plots below, determine whether it's a valid probability density function (pdf).



Interpretation of a pdf

Suppose f is the pdf of X. If f is continuous at a point x, then for small δ

$$P\left(x - \frac{\delta}{2} < X \leq x + \frac{\delta}{2}\right) = \int_{x-\delta/2}^{x+\delta/2} f(u)\,du = \delta f(x).$$

- Is the pdf f of a random variable always ≤ 1?

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  Those that are 6.00001 or 5.99999 feet tall don't count.

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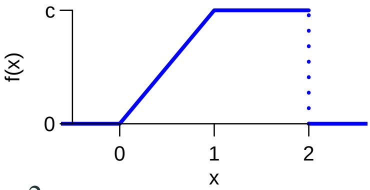
  Those that are 6.00001 or 5.99999 feet tall don't count.
- A pdf $f(x)$ may not be continuous

Example 1

Consider a continuous random variable X with the pdf

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le 1 \\ c & \text{if } 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$



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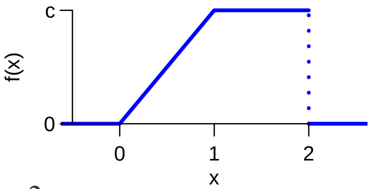

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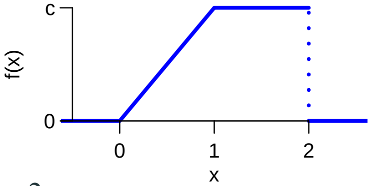

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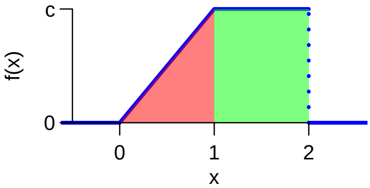
Total Area = Red + Green

$$= \frac{1 \cdot c}{2} + 1 \cdot c = \frac{3}{2}c = 1$$

$$\Rightarrow \quad c = \frac{2}{3}$$



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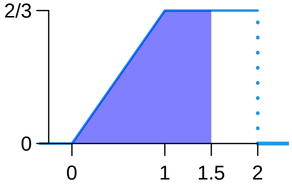

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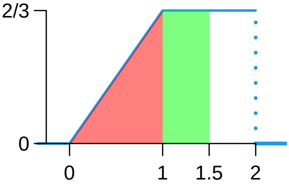
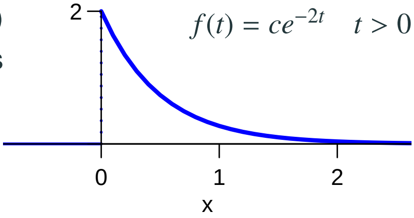


$= \text{Red} + \text{Green}$

$$= \frac{1 \cdot (2/3)}{2} + (0.5)\frac{2}{3} = \frac{2}{3}$$

Example 2

Suppose the lifetime T (in days) of a certain type of batteries has the pdf shown on the right.



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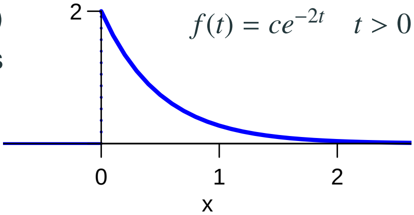

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$$\int_{-\infty}^{\infty} f(t)dt = \int_{0}^{\infty} ce^{-2t}dt = -\frac{c}{2}e^{-2t}\Big|_{t=0}^{t=\infty} = \frac{c}{2} - 0 = 1$$

So $c = 2!$

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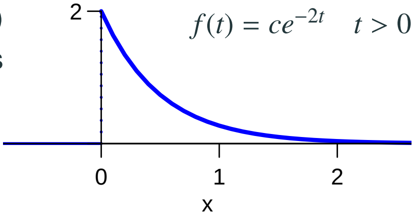

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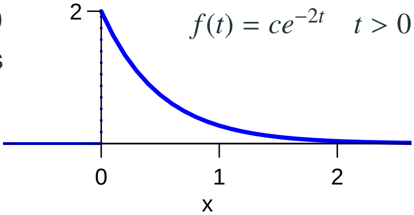

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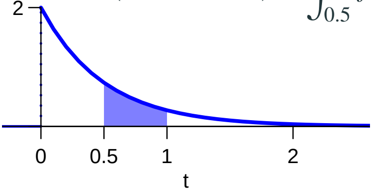

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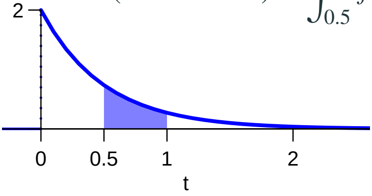

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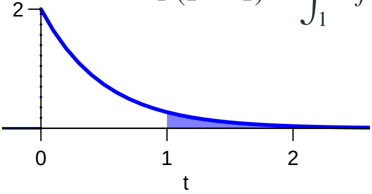

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