STAT 234 Lecture 4
Expected Values of Discrete Random Variables
Section 3.3

Yibi Huang
Department of Statistics
University of Chicago

Random Variable & Probability Mass Function (Review)

A **random variable** is a real-valued function on the sample space S and maps elements of S, ω, to real numbers.

$$\begin{array}{ccc} S & \xrightarrow{X} & \mathbb{R} \\ \omega & \longmapsto & x = X(\omega) \end{array}$$

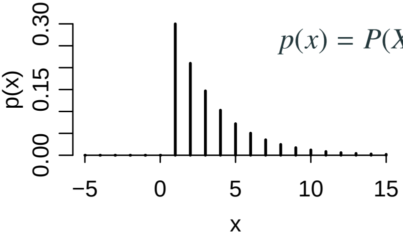
The probability mass function (pmf) of a random variable X is a function $p(x)$ that maps each possible value x_i to the corresponding probability $P(X = x_i)$.

- A pmf $p(x)$ must satisfy $0 \leq p(x) \leq 1$ and $\sum_x p(x) = 1$.

Example: Geometric Distribution

Let X be the number of tosses required to obtain the first heads, when tossing a coin with a probability of p to land heads.

The pmf of X is The pmf of X is



$$
\begin{aligned}
p(x) = P(X = x) &= P(\overbrace{T \ldots T}^{x-1 \text{ tails}} H) \quad \text{by indep.} \\
&= P(T)(T) \cdots P(T)P(H) \\
&= \underbrace{(1-p)(1-p)\cdots(1-p)}_{x-1 \text{ copies}} p \\
&= (1-p)^{x-1} p,
\end{aligned}
$$

if x is a positive integer and $p(x) = 0$ if not.

- We say X has a **geometric distribution** since the pmf is a geometric sequence
- Does $\sum_{x=1}^{\infty} p(x) = 1$?

Example: Geometric Distribution (Cont'd)

Does $\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty}(1-p)^{x-1}p = 1$?

Example: Geometric Distribution (Cont'd)

Does $\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty}(1-p)^{x-1}p = 1$?

Recall the geometric sum

$$\begin{aligned}\sum_{k=0}^{\infty} ar^k &= a + ar + ar^2 + \cdots ar^k + \cdots \\ &= \frac{a}{1-r} \quad \text{if } |r| < 1.\end{aligned}$$

Example: Geometric Distribution (Cont'd)

Does $\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty}(1-p)^{x-1}p = 1$?

Recall the geometric sum

$$\begin{aligned}\sum_{k=0}^{\infty} ar^k &= a + ar + ar^2 + \cdots ar^k + \cdots \\ &= \frac{a}{1-r} \quad \text{if } |r| < 1.\end{aligned}$$

The sum of the pmf of the geometric distribution

$$\begin{aligned}\sum_{x=1}^{\infty} p(x) &= \sum_{x=1}^{\infty}(1-p)^{x-1}p \\ &= p + (1-p)p + (1-p)^2 p + \cdots + (1-p)^{x-1}p + \cdots\end{aligned}$$

is simply the case that $a = p$ and $r = 1 - p$ and hence the sum is

$$\frac{a}{1-r} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1.$$

Example: A Card Game

Consider a card game that you draw ONE card from a well-shuffled deck of cards. You win

- $1 if you draw a heart,
- $5 if you draw an ace (including the ace of hearts),
- $10 if you draw the king of spades and
- $0 for any other card you draw.

What's the pmf of your reward X?

Example: A Card Game

Consider a card game that you draw ONE card from a well-shuffled deck of cards. You win

- \$1 if you draw a heart,
- \$5 if you draw an ace (including the ace of hearts),
- \$10 if you draw the king of spades and
- \$0 for any other card you draw.

What's the pmf of your reward X?

Outcome	x	$p(x)$
Heart (not ace)	1	$12/52$
Ace	5	$4/52$
King of spades	10	$1/52$
All else	0	$35/52$

$$\Rightarrow \quad p(x) = \begin{cases} 35/52 & \text{if } x = 0 \\ 12/52 & \text{if } x = 1 \\ 4/52 & \text{if } x = 5 \\ 1/52 & \text{if } x = 10 \\ 0 & \text{for all other values of } x \end{cases}$$

Expected Value of a Random Variable

Expected Value = Expectation = Mean

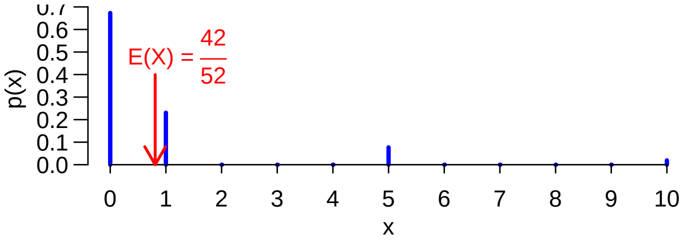
Let X be a discrete random variable with pmf $p(x)$. The **expected value** or the **expectation** or the **mean** of X, denoted by $\mathrm{E}[X]$, or μ is a *weighted average* of the possible values of X, where the weights are the probabilities of those values.

$$\begin{aligned}\mu &= \mathrm{E}[X] \\ &= \sum_{\text{all } x} xP(X = x) \\ &= \sum_{\text{all } x} xp(x)\end{aligned}$$

Example: Card Game — Expected Value

$$p(x) = \begin{cases} 35/52 & \text{if } x = 0 \\ 12/52 & \text{if } x = 1 \\ 4/52 & \text{if } x = 5 \\ 1/52 & \text{if } x = 10 \\ 0 & \text{if } x \neq 0, 1, 5, 10 \end{cases}$$

$$\mathrm{E}[X] = \sum_x x p(x) = 0 \times \frac{35}{52} + 1 \times \frac{12}{52} + 5 \times \frac{4}{52} + 10 \times \frac{1}{52} = \frac{42}{52} \approx 0.81$$



Interpretation of the Expected Value

If one plays the card game 5200 times(where the card is drawn with replacement),then in the 5200 games, he is expected to get

- \$10 about 100 times (why?)
- \$5 about 400 times
- \$1 about 1200 times
- \$0 about 3500 times

His average reward in the 5200 games is hence about

$$\frac{100 \times \$10 + 400 \times \$5 + 1200 \times \$1 + 3500 \times \$0}{5200}$$
$$= \frac{100}{5200} \times \$10 + \frac{400}{5200} \times \$5 + \frac{1200}{5200} \times \$1 + \frac{3500}{5200} \times \$0$$
$$= \frac{1}{52} \times \$10 + \frac{4}{52} \times \$5 + \frac{12}{52} \times \$1 + \frac{35}{52} \times \$0 = \sum_x p(x)x = \$\frac{42}{52} \approx \$0.81$$

So the *long run average reward in a game* is just the **expected value.**

Fair Game

For the card game we have discussed so far,

- will you play the game if it costs $1 to play once?
- will you play the game if it costs 50 cents to play once?
- what is the maximum amount you would be willing to pay to play this game?

Fair Game

For the card game we have discussed so far,

- will you play the game if it costs \$1 to play once?
- will you play the game if it costs 50 cents to play once?
- what is the maximum amount you would be willing to pay to play this game?

A **fair game** is defined as a game that costs as much as its expected payout, i.e. expected profit is 0.

Expected Value of the Geometric Distribution

Recall the pmf of the Geometric distribution is $p(x) = (1-p)^{x-1}p$ for $x = 1, 2, 3, \ldots$. Find the expected value

$$\sum_x xp(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1}p.$$

Sol. Recall the geometric sum

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad \text{if } |r| < 1.$$

Differentiate both sides of the identity above with respect to r, we get another identity

$$\begin{array}{ccc} \dfrac{d}{dr}\sum_{k=0}^{\infty} ar^k & = & \dfrac{d}{dr}\dfrac{a}{1-r} \\ \| & & \| \\ \sum_{k=1}^{\infty} akr^{k-1} & = & \dfrac{a}{(1-r)^2} \end{array}$$

Expected Value of the Geometric Distribution (Cont'd)

Observe the expected value

$$\begin{aligned} \mathrm{E}(X) &= \sum_{x=0}^{\infty} xp(x) = \sum_{x=1}^{\infty} xp(x) \quad \begin{pmatrix} \text{can ignore } x = 0 \text{ since} \\ xp(x) = 0 \text{ when } x = 0 \end{pmatrix} \\ &= \sum_{x=1}^{\infty} x(1-p)^{x-1} p \end{aligned}$$

is simply the second identity above when $a = p$ and $r = 1 - p$, and hence the expected value is

$$\frac{a}{(1-r)^2} = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}.$$

Expected Value of a Function of a Random Variable

Functions of a Random Variable

Example 1 (Card Game w/ Tax). Suppose it costs 50 cents = \$0.5 to play the game and the player has to pay 10% of the reward as tax. One's net profit from the game is

Outcome	Reward X
Heart (not ace)	1
Ace	5
King of spades	10
All else	0

$$\underset{\text{(reward)}}{X} - \underset{\text{(tax)}}{0.1X} - \underset{\text{(cost)}}{0.5}$$

The net profit of the game is hence $h(X) = 0.9X - 0.5$.

Example 2. (Card Game w/ a new Tax Rule) Suppose the tax is $0.02X^2$ dollars for a reward of X dollars. (So those who earn more pay a higher percentage of their rewards as tax). Then the next profit is

$$h(X) = X - 0.02X^2 - 0.5.$$

Expected Value of a Function of a Random Variable

In addition to the expected value of a random variable X itself, we might be also interested in the *expected value of a function of a random variable $h(X)$*, e.g.,

- the net profit from the card game $h(X) = 0.9X - 0.5$
- the net profit from the card game $h(X) = X - 0.02X^2 - 0.5$ with a new tax rule

Definition: If the pmf of X is $p_X(x)$, the expected value of $h(X)$ is

$$\mathrm{E}[h(X)] = \sum_x h(x) p_X(x).$$

Example 1 (Card Game w/ Tax)

One's expected net profit from the game is

Reward x	pmf $p(x)$	Net Profit $h(x) = 0.9x - 0.5$
1	12/52	$0.9 \cdot 1 - 0.5 = 0.4$
5	4/52	$0.9 \cdot 5 - 0.5 = 4.0$
10	1/52	$0.9 \cdot 10 - 0.5 = 8.5$
0	35/52	$0.9 \cdot 0 - 0.5 = -0.5$

$$
\begin{aligned}
\mathrm{E}[h(X)] &= \sum_x h(x)p(x) \\
&= 0.4 \times \frac{12}{52} + 4.0 \times \frac{4}{52} + 8.5 \times \frac{1}{52} + (-0.5) \times \frac{35}{52} \\
&= \frac{11.8}{52} \approx 0.227
\end{aligned}
$$

Variance of a Random Variable

Variance of a Random Variable

One measure of spread of a random variable (or its probability distribution) is the *variance*.

The **variance** of a random variable X, denoted as σ_X^2 or $V(X)$ is defined as the *average squared distance from the mean*.

$$\text{Var}(X) = \sigma^2 = \text{"sigma squared"} = \text{E}\left[(X - \mu)^2\right]$$

Variance is in squared units.
Square root of the variance is the *standard deviation (SD).*

$$\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$$

Example (Card Game)

Recall for the card game reward X:

pmf:

x	0	1	5	10
$p(x)$	$\frac{35}{52}$	$\frac{12}{52}$	$\frac{4}{52}$	$\frac{1}{52}$

, and mean $= \mu = \mathrm{E}(X) = \dfrac{42}{52}$.

Its variance is hence,

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}\left[\left(X - \frac{42}{52}\right)^2\right] = \sum_x \left(x - \frac{42}{52}\right)^2 p(x)$$

$$= \left(0 - \frac{42}{52}\right)^2 \cdot \frac{35}{52} + \left(1 - \frac{42}{52}\right)^2 \cdot \frac{12}{52} + \left(5 - \frac{42}{52}\right)^2 \cdot \frac{4}{52} + \left(10 - \frac{42}{52}\right)^2 \cdot \frac{1}{52}$$

$$= \frac{9260}{52^2} \approx 3.42$$

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{9260}{52^2}} \approx \sqrt{3.42} \approx 1.85.$$

Example (Card Game)

Recall for the card game reward X:

pmf:

x	0	1	5	10
$p(x)$	$\frac{35}{52}$	$\frac{12}{52}$	$\frac{4}{52}$	$\frac{1}{52}$

, and mean $= \mu = \mathrm{E}(X) = \dfrac{42}{52}$.

Its variance is hence,

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}\left[\left(X - \frac{42}{52}\right)^2\right] = \sum_x \left(x - \frac{42}{52}\right)^2 p(x)$$

$$= \left(0 - \frac{42}{52}\right)^2 \cdot \frac{35}{52} + \left(1 - \frac{42}{52}\right)^2 \cdot \frac{12}{52} + \left(5 - \frac{42}{52}\right)^2 \cdot \frac{4}{52} + \left(10 - \frac{42}{52}\right)^2 \cdot \frac{1}{52}$$

$$= \frac{9260}{52^2} \approx 3.42$$

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{9260}{52^2}} \approx \sqrt{3.42} \approx 1.85.$$

Observe the computation of the variance can be awkward if the expected value μ is not an integer.

A Shortcut Formula for Calculating Variance

$$\text{Var}(X) = \text{E}[(X - \mu)^2] = \text{E}(X^2) - \mu^2$$

Proof.

$$\text{E}[(X - \mu)^2] = \sum_x (x - \mu)^2 \, p(x)$$

$$=$$

$$=$$

$$= \qquad\qquad =$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$
\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum\nolimits_x (x-\mu)^2\, p(x) \\
&= \sum\nolimits_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\qquad\qquad}_{} \qquad \underbrace{\qquad\qquad}_{} \qquad \underbrace{\qquad\qquad}_{} \\
&= \qquad\qquad\qquad =
\end{aligned}
$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum_x (x-\mu)^2\, p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum_x x^2\, p(x)}_{=} - 2\mu \underbrace{\sum_x x\, p(x)}_{=} + \mu^2 \underbrace{\sum_x p(x)}
\end{aligned}$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum_x (x-\mu)^2\, p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum_x x\, p(x)} + \mu^2 \underbrace{\sum_x p(x)} \\
&= \qquad\qquad =
\end{aligned}$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum\nolimits_x (x-\mu)^2\, p(x) \\
&= \sum\nolimits_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum\nolimits_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum\nolimits_x x\, p(x)}_{=\mu} + \mu^2 \underbrace{\sum\nolimits_x p(x)} \\
&= \qquad\qquad\qquad =
\end{aligned}$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum\nolimits_x (x-\mu)^2\, p(x) \\
&= \sum\nolimits_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum\nolimits_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum\nolimits_x x\, p(x)}_{=\mu} + \mu^2 \underbrace{\sum\nolimits_x p(x)}_{=1} \\
&= \qquad\qquad =
\end{aligned}$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum\nolimits_x (x-\mu)^2\, p(x) \\
&= \sum\nolimits_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum\nolimits_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum\nolimits_x x\, p(x)}_{=\mu} + \mu^2 \underbrace{\sum\nolimits_x p(x)}_{=1} \\
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 =
\end{aligned}$$

A Shortcut Formula for Calculating Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \sum_x (x-\mu)^2\, p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x) \\
&= \underbrace{\sum_x x^2\, p(x)}_{=\mathrm{E}(X^2)} - 2\mu \underbrace{\sum_x x\, p(x)}_{=\mu} + \mu^2 \underbrace{\sum_x p(x)}_{=1} \\
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 = \mathrm{E}(X^2) - \mu^2
\end{aligned}$$

Example (Card Game)

x	0	1	5	10
$p(x)$	35/52	12/52	4/52	1/52

Let's calculate the variance again using the shortcut formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$. First we calculate $\text{E}[X^2]$

$$\text{E}[X^2] = 0^2 \cdot \frac{35}{52} + 1^2 \cdot \frac{12}{52} + 5^2 \cdot \frac{4}{52} + 10^2 \cdot \frac{1}{52} = \frac{212}{52}$$

and the variance is hence

$$\text{Var}(X) = \text{E}(X^2) - \mu^2 = \frac{212}{52} - \left(\frac{42}{52}\right)^2 = \frac{9260}{52^2}$$

which resembles our previous calculation.

Linear Transformation of a Random Variable

Linear Transformation of a Random Variable

Linear transformation of a random variable $h(X) = aX + b$ is also a function of interest, e.g.,

- The net profit $h(X) = X - 0.1X - 0.5 = 0.9X - 0.5$ from the Card Game w/ tax

For $Y = aX + b$, we can show that

$$\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b, \quad \text{and} \quad \mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

Before we get to the proofs.
Let's review properties of summation.

Review: Summation Notation and Its Properties

In the following, a is a fixed constant.

$$\sum_{i=1}^{n} a = (\underbrace{a + a + \cdots + a}_{n \text{ copies}}) = na$$

$$\begin{aligned}
\sum_{i=1}^{n} (ax_i) &= ax_1 + ax_2 + \cdots + ax_n \\
&= a(x_1 + x_2 + \cdots + x_n) \\
&= a \sum_{i=1}^{n} x_i
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} (x_i + y_i) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) \\
&= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\
&= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i
\end{aligned}$$

Proof of $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

We prove it for the case that X is discrete with pmf $p(x)$.\ This relation is also true when X is continuous.

$$
\begin{aligned}
&\mathrm{E}(aX + b) \\
&= \sum\nolimits_x (ax + b)p(x) && \text{(definition of } \mathrm{E}(aX + b)) \\
&= \sum\nolimits_x (axp(x) + bp(x)) \\
&= \sum\nolimits_x axp(x) + \sum\nolimits_x bp(x) && \text{(since } \sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i) \\
&= a \underbrace{\sum\nolimits_x xp(x)}_{=\mathrm{E}(X)} + b \underbrace{\sum\nolimits_x p(x)}_{=1} && \text{(since } \sum_{i=1}^{n}(ax_i) = a\sum_{i=1}^{n} x_i) \\
&= aE(X) + b
\end{aligned}
$$

Proof of $\text{Var}(aX + b) = a^2 \text{Var}(X)$

Recall $\text{Var}(Y)$ is the expected value of $[Y - \text{E}(Y)]^2$.

Proof of $\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X)$

Recall $\mathrm{Var}(Y)$ is the expected value of $[Y - \mathrm{E}(Y)]^2$.

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = E(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Proof of $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$

Recall $\mathrm{Var}(Y)$ is the expected value of $[Y - \mathrm{E}(Y)]^2$.

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = E(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2$.

Taking expected value of the above we get

$$\mathrm{E}[Y - \mathrm{E}(Y)]^2 \quad = \quad E[a^2(X - \mu)^2]$$

Proof of $\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X)$

Recall $\mathrm{Var}(Y)$ is the expected value of $[Y - \mathrm{E}(Y)]^2$.

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = E(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Taking expected value of the above we get

$$\begin{array}{ccc} \mathrm{E}[Y - \mathrm{E}(Y)]^2 & = & E[a^2(X - \mu)^2] \\ \| & & \|^* \\ \mathrm{Var}(Y) & & a^2 \,\mathrm{E}[(X - \mu)^2] \end{array}$$

Proof of $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Recall $\mathrm{Var}(Y)$ is the expected value of $[Y - \mathrm{E}(Y)]^2$.

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = E(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Taking expected value of the above we get

$$\begin{array}{ccc} \mathrm{E}[Y - \mathrm{E}(Y)]^2 & = & E[a^2(X - \mu)^2] \\ \| & & \|^* \\ \mathrm{Var}(Y) & & a^2\,\mathrm{E}[(X - \mu)^2] \end{array}$$

in which the step $\mathrm{E}[a^2(X - \mu)^2] = a^2\,\mathrm{E}[(X - \mathrm{E}(X))^2]$ is justified using $\mathrm{E}[cW + d] = c\,\mathrm{E}[W] + d$ we just proved with $c = a^2$, $W = (X - E(X))^2$, and $d = 0$.

Proof of $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Recall $\mathrm{Var}(Y)$ is the expected value of $[Y - \mathrm{E}(Y)]^2$.

For $Y = aX + b$, we have proved that $\mathrm{E}(Y) = E(aX + b) = a\mu + b$, where $\mu = \mathrm{E}(X)$ and hence

$$[Y - \mathrm{E}(Y)]^2 = [(aX + b) - \mathrm{E}(aX + b)]^2 = [aX + b - (a\mu + b)]^2 = a^2(X - \mu)^2.$$

Taking expected value of the above we get

$$\begin{array}{ccc} \mathrm{E}[Y - \mathrm{E}(Y)]^2 & = & E[a^2(X-\mu)^2] \\ \| & & \|^* \\ \mathrm{Var}(Y) & & a^2\,\mathrm{E}[(X-\mu)^2] \\ \| & & \| \\ \mathrm{Var}(aX+b) & & a^2\,\mathrm{Var}(X) \end{array}$$

in which the step $\mathrm{E}[a^2(X-\mu)^2] = a^2\,\mathrm{E}[(X - \mathrm{E}(X))^2]$ is justified using $\mathrm{E}[cW + d] = c\,\mathrm{E}[W] + d$ we just proved with $c = a^2$, $W = (X - E(X))^2$, and $d = 0$.

Example (Card Game w/ Tax)

For the Card Game, recall the mean and variance of the reward X are

$$\mathrm{E}(X) = \frac{42}{52}, \quad \mathrm{Var}(X) = \frac{9620}{52^2}$$

The mean and variance of the net profit with tax $h(X) = 0.9X - 0.5$ are

$$\mathrm{E}(0.9X - 0.5) = 0.9\,\mathrm{E}(X) - 0.5 = 0.9 \times \frac{42}{52} - 0.5 = \frac{11.8}{52}$$

$$\mathrm{Var}(0.9X - 0.5) = 0.9^2\,\mathrm{Var}(X) = 0.9^2 \times \frac{9620}{52^2} = \frac{7792.2}{52}$$