STAT 234 Lecture 2
Probability Axioms and Rules,
Conditional Probabilty, Independence

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Outline

Coverage: Section 2.2, 2.4 and 2.5 of MMSA (Skip Section 2.3)

- Probability Axioms
- Complementation Rule
- General Addition Rule
- Conditional Probability
- General Multiplication Rule
- Independence

Probability Axioms and Rules

Probability Axioms

AXIOM 1 For any event A, $P(A) \geq 0$.

AXIOM 2 $P(S) = 1$, where S = sample space.

AXIOM 3 If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Complement Rule

$$\boxed{P(A^c) = 1 - P(A)}$$

Proof. As A and A^c are disjoint and $A \cup A^c = S$ = sample space, by Axiom 3, we have

$$P(S) = P(A) + P(A^c).$$

The Complement Rule then follows from simple algebra and Axiom 2.

$$\begin{aligned} P(A^c) &= P(S) - P(A) && \text{simple algebra} \\ &= 1 - P(A) && \text{by Axiom 2 } P(S) = 1 \end{aligned}$$

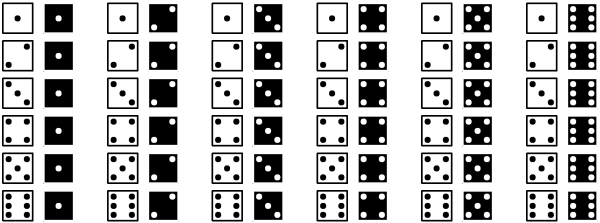
Complement Rule

- Useful for finding probability of events like {at least k} since

$$\{\text{at least } k\}^c = \{\text{at most } k-1\}$$

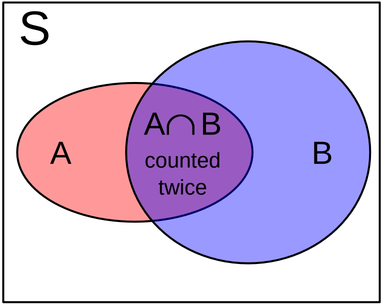
- Example: Rolling a pair of fair dice,

$$\begin{aligned} P(\text{Total is at least 3}) &= 1 - P(\text{Total is 2}) \\ &= 1 - P(\text{The outcome is (1,1)}) \\ &= 1 - \frac{1}{36} \end{aligned}$$



General Addition Rule

$$P(A \cup B) = \begin{cases} P(A) + P(B) & \text{if } A \text{ and } B \text{ are disjoint} \\ P(A) + P(B) - P(A \cap B) & \text{in general} \end{cases}$$



Example (General Addition Rule)

Rolling a pair of fair dice, what is the probability of getting a total of 10 or a double?

Sol. The two events are

$$A = \{\text{Total of 10}\} = \{(4,6),(6,4),(5,5)\} \text{ and}$$

$$B = \{\text{Double}\} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

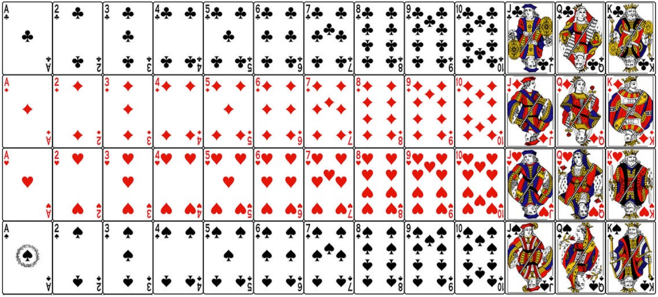
Their intersection is $A \cap B = \{(5,5)\}$.

$$\begin{aligned} &P(\text{Total of 10} \cup \text{double}) \\ &\quad = P(\text{Total of 10}) + P(\text{Double}) - P(\text{Total of 10} \cap \text{double}) \\ &\quad = \frac{3}{36} + \frac{6}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9} \end{aligned}$$

Conditional Probability

Example – Poker

A card is drawn from a well-shuffled deck.



- What is the probability that the card is a King (K)?
  $P(\text{got a King}) = \frac{4}{52}$.
- If the card drawn is known to be a face card (J, Q, K), what is the probability that it is a K?

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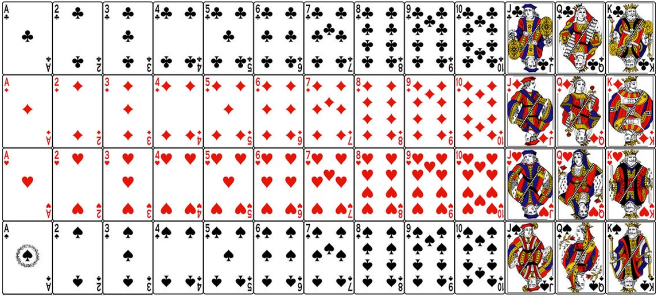

- What is the probability that the card is a King (K)?
  $P(\text{got a King}) = \dfrac{4}{52}$.
- If the card drawn is known to be a face card (J, Q, K), what is the probability that it is a K? $\dfrac{4}{12} = \dfrac{1}{3}$

Conditional Probabilities

Given two events A and B. We denote the probability of event A happens **given** that event B is known to happen as

$$P(A \mid B),$$

read as the probability of "A *given* B."

For the example on the previous slide, let

$$A = \text{the card is a King (K)},$$
$$B = \text{the card is a face card (J,Q,K)}.$$

We have

$$P(A \mid B) = \frac{4}{12} \neq P(A) = \frac{4}{52}.$$

Current information (face card) has changed (restricted) the sample space (possible outcomes).

Exercise

In the previous example, what is

$P(B \,|\, A^c) = P$(the card is a face card | the card is not a King)?

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$P(B \,|\, A^c) = P($the card is a face card | the card is not a King$)$?

- *If the card is not a King, as there are 4 Kings in a deck, then the card must be one of the remaining 48, among which 8 are face cards (4 Jacks and 4 Queens). So the answer is* $8/48$.

Exercise

In the previous example, what is

$P(B \mid A^c) = P($the card is a face card | the card is not a King$)$?

- *If the card is not a King, as there are 4 Kings in a deck, then the card must be one of the remaining 48, among which 8 are face cards (4 Jacks and 4 Queens). So the answer is* $8/48$.

$P(B \mid A) = P($the card is a face card | the card is a King$)$?

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$P(B \mid A) = P($the card is a face card | the card is a King$)$?

- $= 1$, *since a King is a face card.*

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- $= 1$, *since a King is a face card.*

$P(A \mid B^c) = P($the card is a King | the card is not a face card$)$?

Exercise

In the previous example, what is

$P(B \mid A^c) = P(\text{the card is a face card} \mid \text{the card is not a King})$?

- *If the card is not a King, as there are 4 Kings in a deck, then the card must be one of the remaining 48, among which 8 are face cards (4 Jacks and 4 Queens). So the answer is* $8/48$.

$P(B \mid A) = P(\text{the card is a face card} \mid \text{the card is a King})$?

- $= 1$, *since a King is a face card.*

$P(A \mid B^c) = P(\text{the card is a King} \mid \text{the card is not a face card})$?

- $= 0$, *since a King is a face card. If it's not a face card, it cannot be a King.*

Definition of Conditional Probability

The MMSA textbook defines the **conditional probability** $P(A \mid B)$ as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

Example.

- P(face card) $= 12/52$
- P(face card $\cap$ King) = P(King) $= 4/52$

By definition of conditional probability,

$$P(\text{King} \mid \text{face card}) = \frac{P(\text{face card} \cap \text{King})}{P(\text{face card})} = \frac{4/52}{12/52} = \frac{4}{12}$$

Example (Credit Cards)

Consider randomly selecting a student at a certain university, and let

- V = the event that the selected student has a Visa credit card
- M = the analogous event for a MasterCard.

Suppose that $P(V) = 0.5$, $P(M) = 0.4$, and $P(V \cap M) = 0.25$.

Interpret and calculate the probability $P(V \mid M)$

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$P(V \mid M) =$ the proportion of Visa card owners among those who already have a MasterCard.

$$P(V \mid M) = \frac{P(V \cap M)}{P(M)} = \frac{0.25}{0.4} = 0.625.$$

Example (Credit Cards, Cont'd)

Interpret and calculate the probability $P(V \mid M^c)$

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Interpret and calculate the probability $P(V \mid M^c)$

$P(V \mid M^c)$ = the proportion of Visa card owners among those who have no MasterCard.

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$P(V \mid M^c)$ = the proportion of Visa card owners among those who have no MasterCard.

$$P(V \mid M^c) = \frac{P(V \cap M^c)}{P(M^c)}$$ by definition of cond. prob.

Example (Credit Cards, Cont'd)

Interpret and calculate the probability $P(V \mid M^c)$

$P(V \mid M^c)$ = the proportion of Visa card owners among those who have no MasterCard.

$$P(V \mid M^c) = \frac{P(V \cap M^c)}{P(M^c)} \qquad \text{by definition of cond. prob.}$$

$$= \frac{P(V \cap M^c)}{1 - P(M)} \qquad \text{complement rule}$$

Example (Credit Cards, Cont'd)

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\begin{aligned}
P(V \mid M^c) &= \frac{P(V \cap M^c)}{P(M^c)} && \text{by definition of cond. prob.} \\
&= \frac{P(V \cap M^c)}{1 - P(M)} && \text{complement rule} \\
&= \frac{P(V) - P(V \cap M)}{1 - P(M)} && \text{Venn Diagram}
\end{aligned}
$$

Example (Credit Cards, Cont'd)

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Calculation of Conditional Probabilities

Do NOT always calculate conditional probabilities by the definition .

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Sometimes one can calculate $P(A \mid B)$ by thinking about how B has changed the sample space instead of finding $P(A \cap B)$ and $P(B)$ and calculating their ratio.

Example. A deck of cards is well-shuffled and the two cards are drawn w/o replacement. What is the probability that second card is a King

- given that the first card is a King?

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- given that the first card is a King? $3/51$

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- given that the first card is NOT a King?

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- given that the first card is a King? $3/51$
- given that the first card is NOT a King? $4/51$

Example (Age and Rank of Professors)

The table below cross-classifies faculty members in a certain university by age and rank.

Age (year)	Full professor	Associate professor	Assistant professor	Lecturer	**Total**
Under 40	54	173	220	23	470
40-49	156	125	61	6	348
50-59	145	68	36	4	253
60+	75	15	3	0	93
Total	430	381	320	33	1164

Find P(full prof. | under 40)

- by restricting the sample space;
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Find $P(\text{full prof.} \mid \text{under 40})$

- by restricting the sample space; 54/470
- by definition.

$$P(\text{full prof.} \mid \text{under 40}) = \frac{P(\text{full prof.} \cap \text{ under 40})}{P(\text{under 40})} = \frac{54/1164}{470/1164} = \frac{54}{470}$$

General Multiplication Rule

General Multiplication Rule

The formula for conditional probability

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

can be used the other way around. Multiplying both sides by $P(A)$, we get the *General Multiplication Rule*:

$$\boxed{P(A \cap B) = P(A) \times P(B \mid A)}$$

If we want $P(A \cap B)$, and both $P(A)$, $P(B \mid A)$ are known or are easy to compute, we can use the General Multiplication Rule.

Example: General Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that both cards are Kings?

Solution. Let

$$A = \text{ 1st card is a King,}$$
$$B = \text{ 2nd card is a King.}$$

- $P(A) = P(\text{the 1st card is a King}) =$.
- Given that the 1st card is a King, the conditional probability that the 2nd card is a King $=?$.
- So the probability that both cards are Kings $=?$

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- Given that the 1st card is a King, the conditional probability that the 2nd card is a King $=?$ $P(B\,|\,A) = \dfrac{3}{51}$.
- So the probability that both cards are Kings $=?$

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- $P(A) = P(\text{the 1st card is a King}) = 4/52$.
- Given that the 1st card is a King, the conditional probability that the 2nd card is a King =? $P(B|A) = \frac{3}{51}$.
- So the probability that both cards are Kings =?

$$P(A \cap B) = P(A) \times P(B|A) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221} \approx 0.0045.$$

Example: General Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that neither card is a K?

Sol.

- $P(A^c) = P(\text{the 1st card is not a K}) =$.
- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a K $=?$.
- So the probability that neither card is a K $=?$
- P(at least one of the two cards is a K) $=?$

Example: General Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that neither card is a K?

Sol.

- $P(A^c) = P(\text{the 1st card is not a K}) = 48/52$.
- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a K $= ?$.
- So the probability that neither card is a K $= ?$
- P(at least one of the two cards is a K) $= ?$

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- $P(A^c) = P(\text{the 1st card is not a K}) = 48/52$.
- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a K $=?$ $P(B^c \,|\, A^c) = \frac{47}{51}$.
- So the probability that neither card is a K $=?$

- P(at least one of the two cards is a K) $=?$

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- $P(A^c) = P(\text{the 1st card is not a K}) = 48/52$.
- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a K $=?$ $P(B^c \mid A^c) = \frac{47}{51}$.
- So the probability that neither card is a K $=?$ $P(A^c \cap B^c) = P(A^c) \times P(B^c \mid A^c) = \frac{48}{52} \times \frac{47}{51} = \frac{188}{221} \approx 0.851.$
- P(at least one of the two cards is a K) $=?$

Example: General Multiplication Rule

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Sol.

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- Given that the 1st card is not a K, the conditional probability that the 2nd card is not a K =? $P(B^c \mid A^c) = \dfrac{47}{51}$.
- So the probability that neither card is a K =?
  $P(A^c \cap B^c) = P(A^c) \times P(B^c \mid A^c) = \dfrac{48}{52} \times \dfrac{47}{51} = \dfrac{188}{221} \approx 0.851.$
- P(at least one of the two cards is a K) =?

{at least a K}^{c} ={neither is a K}

So, P(at least a K)= $1-$ P(neither is K) $= 1 - 0.851 = 0.149.$

General Multiplication Rule for Several Events

$$P(ABC) = P(A) \times P(B|A) \times P(C|AB)$$
$$P(ABCD) = P(A) \times P(B|A) \times P(C|AB) \times P(D|ABC)$$
$$P(ABCDE) = P(A) \times P(B|A) \times P(C|AB) \times P(D|ABC) \times P(E|ABCD)$$

and so on

Example: General Multiplication Rule for Several Events

Five cards are dealt from a deck of well-shuffled card. What is the chance that none of them are hearts $\heartsuit$?

Sol. Let A_i be the event that the ith card dealt is not a $\heartsuit$.

- $P(A_1) = P(\text{1st card is not a } \heartsuit) = 39/52$

Example: General Multiplication Rule for Several Events

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- $P(A_1) = P(\text{1st card is not a } \heartsuit) = 39/52$
- Given the 1st card is not a $\heartsuit$, the conditional probability that the 2nd is not a $\heartsuit$ $= P(A_2|A_1) = \dfrac{38}{51}$.

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- Given the 1st card is not a $\heartsuit$, the conditional probability that the 2nd is not a $\heartsuit = P(A_2|A_1) = \dfrac{38}{51}$.
- Given neither of the first two cards is a $\heartsuit$, the condition probability that the 3rd is not a $\heartsuit = P(A_3|A_1A_2) = \dfrac{37}{50}$.

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- Given neither of the first two cards is a $\heartsuit$, the condition probability that the 3rd is not a $\heartsuit = P(A_3|A_1A_2) = \dfrac{37}{50}$.
- Likewise, $P(A_4|A_1A_2A_3) = \dfrac{36}{49}$, $P(A_5|A_1A_2A_3A_4) = \dfrac{35}{48}$

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- Likewise, $P(A_4|A_1A_2A_3) = \dfrac{36}{49}$, $P(A_5|A_1A_2A_3A_4) = \dfrac{35}{48}$
- By the General Multiplication Rule,

$$P(A_1A_2A_3A_4A_5) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50} \times \frac{36}{49} \times \frac{35}{48} \approx 0.222$$

Continue the previous slide, what is the probability of getting at least one heart $\heartsuit$ among the five cards?

Sol. Since {at least one $\heartsuit$}^{c} ={no $\heartsuit$}, by the complement rule,

$$P(\text{at least one } \heartsuit) = 1 - P(\text{no } \heartsuit) = 1 - 0.222 = 0.778.$$

Independence

Independence

Two events A and B are said to be independent if any of the following is true

- $P(A|B) = P(A)$ .... B happens doesn't affect how likely A happens
- $P(A|B) = P(A|B^c)$ .... How likely A happens is not affected by B happens or not
- $P(B|A) = P(B)$ .... A happens doesn't affect how likely B happens
- $P(A \cap B) = P(A) \times P(B)$

If any of the identities above is true, then all remaining identities will also be true.

Proof of $P(A|B) = P(A)$ implies $P(B|A) = P(B)$

$$
\begin{aligned}
P(B|A) &= \frac{P(A \cap B)}{P(A)} && \text{definition of conditional prob.} \\
&= \frac{P(B)P(A|B)}{P(A)} && \text{General multiplication rule} \\
&= \frac{P(B)P(A)}{P(A)} && \text{since } P(A|B) = P(A) \\
&= P(B)
\end{aligned}
$$

Thus, $P(A|B) = P(A)$ implies $P(B|A) = P(B)$.

Proof of $P(B|A) = P(B)$ implies $P(A \cap B) = P(A)P(B)$

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B|A) && \text{(by General multiplication rule)} \\
&= P(A)P(B) && \text{(since } P(B|A) = P(B)\text{)}
\end{aligned}
$$

Practice – Checking Independence

Between January 9-12, 2013, SurveyUSA interviewed a random sample of 500 NC residents asking them whether they think widespread gun ownership protects law abiding citizens from crime, or makes society more dangerous. 58% of all respondents said it protects citizens. 67% of White respondents, 28% of Black respondents, and 64% of Hispanic respondents shared this view. Are opinion on gun ownership and race ethnicity independent?

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P(protects citizens) varies by race/ethnicity, therefore opinion on gun ownership and race ethnicity are dependent.

http://www.surveyusa.com/client/PollReport.aspx?g=a5f460ef-bba9-484b-8579-1101ea26421b

Independent Events vs Disjoint Events

- If A and B are independent, $P(A \cap B) = P(A) \times P(B)$.
- If A and B are disjoint: $A \cap B = \varnothing \Rightarrow P(A \cap B) = 0$.
- If $P(A) > 0$ and $P(B) > 0$,
  - Independent events **cannot** be disjoint.
  - Disjoint events **cannot** be independent.
- Conceptually, A and B are disjoint means that one happens prevents the other from happening, so one's occurrence definitely affects the other's.

Multiplication Rule for Independent Events

When A and B are independent

$$P(A \cap B) = P(A) \times P(B)$$

- This is simply the general multiplication rule: $P(A \cap B) = P(A) \times P(B|A)$ in which $P(B|A)$ reduce to $P(B)$ when A and B are independent
- More generally,

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1) \times P(A_2) \times \cdots \times P(A_k)$$

  if $A_1, \ldots, A_k$ are independent.

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Exercise: You roll a 6-face die twice, what is the probability of getting two aces in a row?

$$P(\text{ace in the 1st roll}) \times P(\text{ace on the 2nd roll}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Abuse of the Multiplication Rule

As estimated in 2020, of the U.S. population,

- 2.0% were 85 or older, and
- 49.5% were male.

True or False and explain: $0.495 \times 0.02 \approx 1\%$ of the U.S. population are males and age 85+.

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True or False and explain: $0.495 \times 0.02 \approx 1\%$ of the U.S. population are males and age 85+.

False, Age and Gender are dependent.
In particular, as women on average live longer than men,
there are more old women than old men.

Among 85+ year-olds, only 36.5% are male, not 49.5%.
Of the U.S. population in 2020, only

$$P(M \cap 85+) = P(85+)P(M|85+) = 0.02 \times 0.365 \approx 0.0073 = 0.73\%$$

were males age 85+.