STAT 222 Lecture 24
Confounded Two-Level Factorial Designs
In 2 Blocks

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Coverage

Section 13.1-13.3 of Dean & Voss

Example 13.3.1 Field Experiment

A 2^4 experiment on the yield of beans where the 4 factors are

- A = amount of dung (0 or 10 tons) spread per acre
- B = amount of nitrochalk (0 and 45 lb) per acre
- C = amount of superphosphate (0 and 67 lb) per acre.
- D = amount of muriate of potash (0 and 112 lb) per acre.

Two dissimilar blocks of land.
Each block was divided into 8 plots.

A **single-replicate** experiment with $2^4 = 16$ factor combinations (TC) divided into $b = 2$ blocks of size $k = 8$.

Incomplete block design but not BIBD

Block I		Block II	
Trtmt	Yield	Trtmt	Yield
0000	58	0001	55
0011	51	0010	45
0101	44	0100	42
0110	50	0111	36
1001	43	1000	53
1010	50	1011	55
1100	41	1101	41
1111	44	1110	48

2^3 Design in Two Blocks

For a single-replicate $2 \times 2 \times 2 = 2^3$ design that each factor has 2 levels (0 = low, 1 = high), the 8 treatments are denoted as

$$000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111.$$

Suppose there are **two blocks each of size 4** available. How to divide the 8 treatments into the two blocks so that as many parameters in the 3-way model below can be estimated as possible?

$$y_{ijkh} = \mu + \underbrace{\alpha_i + \beta_j + \gamma_k + \alpha\beta_{ij} + \beta\gamma_{jk} + \alpha\gamma_{ij} + \alpha\beta\gamma_{ijk}}_{\text{treatment}} + \underbrace{\theta_h}_{\text{block}} + \varepsilon_{ijkh}$$

- 8 observations in total, total df $= 8 - 1 = 7$

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- 7 df for treatments + 1 df for blocks $>$ 7 df in total

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- 8 observations in total, total df $= 8 - 1 = 7$
- 7 df for treatments + 1 df for blocks $>$ 7 df in total
- not all parameters can be estimated

Recall parameter estimates for the full 3-way model of 2^3 design

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \alpha\beta_{ij} + \beta\gamma_{jk} + \alpha\gamma_{ij} + \alpha\beta\gamma_{ijk} + \varepsilon_{ijk},$$

under the zero-sum constraints are of the form $\sum_{ijk} c_{ijk} y_{ijk}/2^3$ where the coefficients c_{ijk} are as shown in the table below.

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC
000	1	−1	−1	−1	1	1	1	−1
001	1	−1	−1	1	1	−1	−1	1
010	1	−1	1	−1	−1	1	−1	1
011	1	−1	1	1	−1	−1	1	−1
100	1	1	−1	−1	−1	−1	1	1
101	1	1	−1	1	−1	1	−1	−1
110	1	1	1	−1	1	−1	−1	−1
111	1	1	1	1	1	1	1	1

For example,

$$\widehat{\mu} = (y_{000} + y_{001} + y_{010} + y_{011} + y_{100} + y_{101} + y_{110} + y_{111})/8$$

$$\widehat{\alpha}_1 = (-y_{000} - y_{001} - y_{010} - y_{011} + y_{100} + y_{101} + y_{110} + y_{111})/8$$

$$\widehat{\alpha\beta}_{11} = (y_{000} + y_{001} - y_{010} - y_{011} - y_{100} - y_{101} + y_{110} + y_{111})/8$$

2^3 Design in 2 Blocks, Confounding ABC

Let's try dividing the treatments by the coefficients of the ABC contrast.

- placing those with coefficient $c_{ijk}^{ABC} = +1$ in one block,
- and those with coefficient $c_{ijk}^{ABC} = -1$ in the other block

	$\widehat{\mu}$	$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\gamma}_1$	$\widehat{\alpha\beta}_{11}$	$\widehat{\alpha\gamma}_{11}$	$\widehat{\beta\gamma}_{11}$	$\widehat{\alpha\beta\gamma}_{111}$	Block	
	(1)	A	B	C	AB	AC	BC	ABC	I	II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1		✓
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1	✓	
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1	✓	
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1		✓

Can one estimate μ, α_1, β_1, γ_1, $\alpha\beta_{11}, \ldots$ etc in this design?

Under the model,

$$y_{ijkh} = \mu + \overbrace{\alpha_i + \beta_j + \gamma_k + \alpha\beta_{ij} + \beta\gamma_{jk} + \alpha\gamma_{ij} + \alpha\beta\gamma_{ijk}}^{\text{treatment}} + \overbrace{\theta_h}^{\text{block}} + \varepsilon_{ijkh}$$

if there is no block effect, $\theta_h = 0$ for both blocks, the model above is simply the full 3-way model. We know the estimates

$$\widehat{\mu},\ \widehat{\alpha}_1,\ \widehat{\beta}_1,\ \widehat{\gamma}_1,\ \widehat{\alpha\beta}_{11},\ \widehat{\alpha\gamma}_{11},\ \widehat{\beta\gamma}_{11},\ \widehat{\alpha\beta\gamma}_{111}$$

defined by their coefficients given in the table below would be unbiased estimates of their corresponding parameters.

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC	Block I	Block II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1		✓
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1	✓	
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1	✓	
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1		✓

We hence just need to check whether the expected values of these estimates are affected by the extra block effect.

Estimating μ in 2^3 Design in 2 Blocks, Confounding ABC

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC	Block I	II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1		✓
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1	✓	
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1	✓	
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1		✓

$\widehat{\mu}$ is the average of the 8 observations of which half are in Block I and half in Block II. Contribution of the block effect to the expected value of $\widehat{\mu}$ is

$$\theta_I + \theta_{II} + \theta_{II} + \theta_I + \theta_{II} + \theta_I + \theta_I + \theta_{II} = 0$$

adds up to 0 because of the zero-sum constraints $\theta_I + \theta_{II}$.

Hence, $\widehat{\mu}$ remains an unbiased estimate for μ even if the block effect is present.

Estimating α_1 in 2^3 Design in 2 Blocks, Confounding ABC

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC	Block I	Block II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1		✓
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1	✓	
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1	✓	
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1		✓

Observe among the 4 observations w/ $c^A_{ijk} = +1$, half are in Block I and half in Block II. Their block effects add up to $\theta_I + \theta_I + \theta_{II} + \theta_{II}$.

Likewise, the 4 observations that $c^A_{ijk} = -1$ also split evenly in the two blocks. The sum of their block effects are $\theta_I + \theta_I + \theta_{II} + \theta_{II}$.

$\widehat{\alpha}_1$ remains unbiased as the Block effect affects its expected value by

$$\underbrace{\theta_I + \theta_I + \theta_{II} + \theta_{II}}_{\text{from those w/ } C^A_{ijk}=1} - \underbrace{(\theta_I + \theta_I + \theta_{II} + \theta_{II})}_{\text{from those w/ } C^A_{ijk}=-1} = 0.$$

ABC Is Confounded w/ Blocks. Other Parameters Can Be Estimated

One can replace c_{ijk}^{A} with the coefficients c_{ijk}^{U} of another contrast U and the argument on the previous page remain valid. Hence, these parameter estimates remain unbiased estimates for their corresponding parameters even if the block effect is present.

ABC Is Confounded w/ Blocks. Other Parameters Can Be Estimated

One can replace c_{ijk}^A with the coefficients c_{ijk}^U of another contrast U and the argument on the previous page remain valid. Hence, these parameter estimates remain unbiased estimates for their corresponding parameters even if the block effect is present.

The only exception is the contrast ABC which we used to define the blocks.

- The 4 observations w/ $c_{ijk}^{ABC} = -1$ are all in Block I, Their block effects add up to $\theta_I + \theta_I + \theta_I + \theta_I$.
- The 4 observations that $c_{ijk}^A = +1$ are all in Block II. The sum of their block effects are $\theta_{II} + \theta_{II} + \theta_{II} + \theta_{II}$.
- The expected value of $\widehat{\alpha\beta}\gamma_{111}$ is affected by block by

$$-\underbrace{(\theta_I + \theta_I + \theta_I + \theta_I)}_{\text{from those w/ } C_{ijk}^{ABC} = -1} + \underbrace{(\theta_{II} + \theta_{II} + \theta_{II} + \theta_{II})}_{\text{from those w/ } C_{ijk}^{ABC} = 1} = 4(\theta_{II} - \theta_I) \neq 0.$$

We hence said the ABC interaction is confounded with block effects and cannot be estimated.

Summary of 2^k Design in 2 Blocks w/ ABC Confounded

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC	Block I	II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1		✓
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1	✓	
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1	✓	
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1		✓

- ABC interaction is confounded w/ block effects and hence cannot be estimated
- All other parameters can be estimated as in a 2^3 design without blocking

2^3 Design in 2 Blocks w/ Other Contrasts Confounded

One can also use other contrasts, like AB, to define the blocks.

	$\widehat{\mu}$ (1)	$\widehat{\alpha}_1$ A	$\widehat{\beta}_1$ B	$\widehat{\gamma}_1$ C	$\widehat{\alpha\beta}_{11}$ AB	$\widehat{\alpha\gamma}_{11}$ AC	$\widehat{\beta\gamma}_{11}$ BC	$\widehat{\alpha\beta\gamma}_{111}$ ABC	Block I	II
000	1	−1	−1	−1	1	1	1	−1	✓	
001	1	−1	−1	1	1	−1	−1	1	✓	
010	1	−1	1	−1	−1	1	−1	1		✓
011	1	−1	1	1	−1	−1	1	−1		✓
100	1	1	−1	−1	−1	−1	1	1		✓
101	1	1	−1	1	−1	1	−1	−1		✓
110	1	1	1	−1	1	−1	−1	−1	✓	
111	1	1	1	1	1	1	1	1	✓	

For such designs, all parameters can be estimated except for the one that is used to define the blocks, which will be confounded with block effects.

- AB interactions would be confounded with blocks in the design above and hence cannot be estimated

2^k Design in 2 Blocks of Size 2^{k-1}

The 2^3 factorial in 2 blocks design can be generalized to 2^k designs.

- 2 blocks of size 2^{k-1}
- The 2 blocks are defined by one of the contrasts, usually the contrast for the highest-order interaction since it's of the least interest.
  - e.g., the Field experiment introduced in the beginning is a 2^4 design in 2 blocks that ABCD is confounded
- If we are certain that some lower order interactions are zero, we can use it to define the blocks
- All parameters can be estimated as in a 2^k design without blocking except for the one that is confounded.

Back to the Field Experiment

TC	$y_{ijk\ell}$	B	BC	$ABCD$
0000	58	-1	1	1
0001	55	-1	1	-1
0010	45	-1	-1	-1
0011	51	-1	-1	1
0100	42	1	-1	-1
0101	44	1	-1	1
0110	50	1	1	1
0111	36	1	1	-1
1000	53	-1	1	-1
1001	43	-1	1	1
1010	50	-1	-1	1
1011	55	-1	-1	-1
1100	41	1	-1	1
1101	41	1	-1	-1
1110	48	1	1	-1
1111	44	1	1	1

Treatments are divided into blocks by the ABCD contrast.

Block	Treatment
I	0000, 0011, 0101, 0110 1001, 1010, 1100, 1111
II	0001, 0010, 0100, 0111 1000, 1011, 1101, 1110

Back to the Field Experiment

TC	$y_{ijk\ell}$	B	BC	$ABCD$
0000	58	−1	1	1
0001	55	−1	1	−1
0010	45	−1	−1	−1
0011	51	−1	−1	1
0100	42	1	−1	−1
0101	44	1	−1	1
0110	50	1	1	1
0111	36	1	1	−1
1000	53	−1	1	−1
1001	43	−1	1	1
1010	50	−1	−1	1
1011	55	−1	−1	−1
1100	41	1	−1	1
1101	41	1	−1	−1
1110	48	1	1	−1
1111	44	1	1	1

$$\begin{aligned}
\widehat{\beta}_1 &= (-y_{0000} - y_{0001} - y_{0010} - y_{0011} \\
&\quad + y_{0100} + y_{0101} + y_{0110} + y_{0111} \\
&\quad - y_{1000} - y_{1001} - y_{1010} - y_{1011} \\
&\quad + y_{1100} + y_{1101} + y_{1110} + y_{1111})/16 \\
&= (-58 - 55 - 45 - 51 \\
&\quad + 42 + 44 + 50 + 36 \\
&\quad - 53 - 43 - 50 - 55 \\
&\quad + 41 + 41 + 48 + 44)/16 \\
&= -4
\end{aligned}$$

$$SS_B = \sum_{ijk\ell} \widehat{\beta}_j^2 = 16\widehat{\beta}_j^2 = 16(-4)^2 = 256$$

Back to the Field Experiment

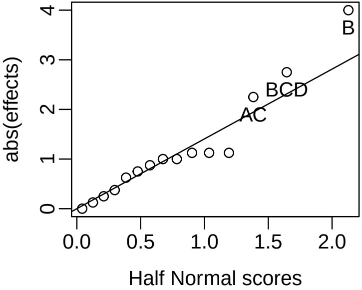
TC	$y_{ijk\ell}$	B	BC	$ABCD$
0000	58	−1	1	1
0001	55	−1	1	−1
0010	45	−1	−1	−1
0011	51	−1	−1	1
0100	42	1	−1	−1
0101	44	1	−1	1
0110	50	1	1	1
0111	36	1	1	−1
1000	53	−1	1	−1
1001	43	−1	1	1
1010	50	−1	−1	1
1011	55	−1	−1	−1
1100	41	1	−1	1
1101	41	1	−1	−1
1110	48	1	1	−1
1111	44	1	1	1

$$
\begin{aligned}
\widehat{\beta\gamma}_{11} &= (y_{0000} + y_{0001} - y_{0010} - y_{0011} \\
&\quad - y_{0100} - y_{0101} + y_{0110} + y_{0111} \\
&\quad + y_{1000} + y_{1001} - y_{1010} - y_{1011} \\
&\quad - y_{1100} - y_{1101} + y_{1110} + y_{1111})/16 \\
&= (58 + 55 - 45 - 51 \\
&\quad - 42 - 44 + 50 + 36 \\
&\quad + 53 + 43 - 50 - 55 \\
&\quad - 41 - 41 + 48 + 44)/16 \\
&= 1.125 \\
SS_{BC} &= \sum_{ijk\ell} \widehat{\beta\gamma}_{jk} 2 = 16(\widehat{\beta\gamma}_{11})^2 \\
&= 16(1.125)^2 = 20.25
\end{aligned}
$$

```r
field = read.table(
  "https://www.stat.uchicago.edu/~yibi/s222/field.txt",h=T)
field$A = as.factor(field$A)
field$B = as.factor(field$B)
field$C = as.factor(field$C)
field$D = as.factor(field$D)
contrasts(field$A) = contr.sum(2)
contrasts(field$B) = contr.sum(2)
contrasts(field$C) = contr.sum(2)
contrasts(field$D) = contr.sum(2)
lmfield = lm(yield ~ block + A*B*C*D, data=field)
lmfield$coef
(Intercept)        block           A1           B1           C1
  4.838e+01   -7.500e-01    3.750e-01    4.000e+00   -1.250e-01
         D1        A1:B1        A1:C1        B1:C1        A1:D1
  1.125e+00    6.250e-01    2.250e+00    1.125e+00    2.012e-16
      B1:D1        C1:D1     A1:B1:C1     A1:B1:D1     A1:C1:D1
 -8.750e-01    2.500e-01    1.000e+00   -1.000e+00   -1.125e+00
   B1:C1:D1  A1:B1:C1:D1
  2.750e+00           NA
```

Half-Normal Probablity Plot of Field Experiment

```
library(daewr)
halfnorm(lmfield$coef[2:16], alpha=0.2)
```

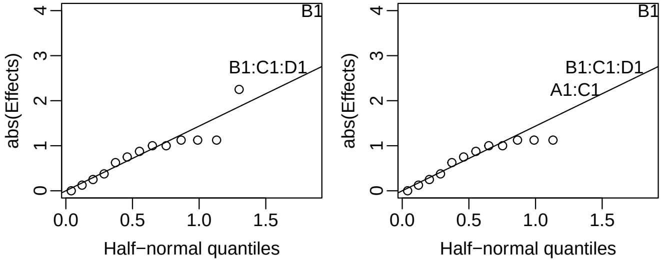



```
zscore= 0.04179 0.1257 0.2104 0.2967 0.3853 0.477 0.573 0.6745 0.7835 0
```

- B main effect is the most prominent one
- BCD and AC interactions might be present, not sure
- All other effects are small
- No p-value is provided in a half-normal plot

The faraway library can also produce the halfnorm() plot if one cannot install the daewr library.

```
library(faraway)
halfnorm(lmfield$coef[-1], labs= names(lmfield$coef[-1]),
         ylab= "abs(Effects)")
qqline(c(-abs(lmfield$coef[-1]),abs(lmfield$coef[-1])))
halfnorm(lmfield$coef[-1], nlab = 3, labs= names(lmfield$coef[-1]),
         ylab= "abs(Effects)")
qqline(c(-abs(lmfield$coef[-1]),abs(lmfield$coef[-1])))
```



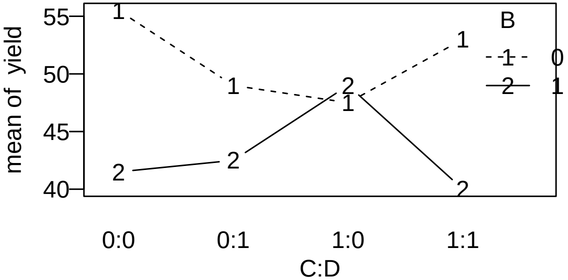
- By default, B and BCD are labelled but AC is not labelled.
- One can specify the number of labelled effects to 3 by adding nlab=3 so the 3rd largest effect AC is labelled.

```
anova(lm(yield ~ block + A*B*C*D, data=field))
Warning in anova.lm(lm(yield ~ block + A * B * C * D, data
= field)): ANOVA F-tests on an essentially perfect fit are
unreliable
Analysis of Variance Table

Response: yield
          Df Sum Sq Mean Sq F value Pr(>F)
block      1   2.25    2.25     NaN    NaN
A          1   2.25    2.25     NaN    NaN
B          1 256.00  256.00     NaN    NaN
C          1   0.25    0.25     NaN    NaN
D          1  20.25   20.25     NaN    NaN
A:B        1   6.25    6.25     NaN    NaN
A:C        1  81.00   81.00     NaN    NaN
B:C        1  20.25   20.25     NaN    NaN
A:D        1   0.00    0.00     NaN    NaN
B:D        1  12.25   12.25     NaN    NaN
C:D        1   1.00    1.00     NaN    NaN
A:B:C      1  16.00   16.00     NaN    NaN
A:B:D      1  16.00   16.00     NaN    NaN
A:C:D      1  20.25   20.25     NaN    NaN
B:C:D      1 121.00  121.00     NaN    NaN
Residuals  0   0.00     NaN
```

BCD Interaction Plots

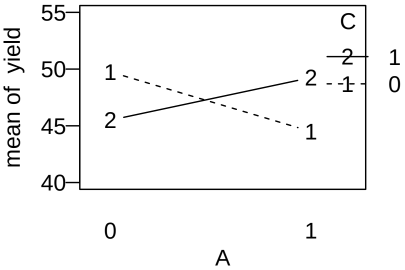
```
par(mai=c(.6,.6,.05,.1),mgp=c(2,.5,0), las=1)
with(field, interaction.plot(C:D, B, yield, type="b"))
```



- yield is lower when B is at high level
  adding nitrochalk decreased the yield, averaged over levels of A, C, and D.
- BD interaction changed w/ C
  - When $C = 0$,
  - When $C = 1$,

AC Interaction Plots

```
par(mai=c(.6,.6,.05,.3),mgp=c(2,.5,0), las=1)
with(field, interaction.plot(A, C, yield, type="b", ylim=c(40,55)))
```



Analysis By Pooling Terms Into Error

Suppose the researcher had known ahead of time that no AD interaction,

- then no ABD, ACD, ABCD interaction either

Analysis By Pooling Terms Into Error

Suppose the researcher had known ahead of time that no AD interaction,

- then no ABD, ACD, ABCD interaction either
- could pool AD, ABD, ACD into error

Analysis By Pooling Terms Into Error

Suppose the researcher had known ahead of time that no AD interaction,

- then no ABD, ACD, ABCD interaction either
- could pool AD, ABD, ACD into error
- couldn't pool ABCD since it's confounded w/ block

Analysis By Pooling Terms Into Error

Suppose the researcher had known ahead of time that no AD interaction,

- then no ABD, ACD, ABCD interaction either
- could pool AD, ABD, ACD into error
- couldn't pool ABCD since it's confounded w/ block

Then

$$\text{SSE} = SS_{AD} + SS_{ABD} + SS_{ACD} = 0 + 16 + 20.25 = 36.25$$
$$\text{MSE} = \text{SSE}/3 \approx 12.0833$$

with dfE $= 3$.

By pooling AD, ABD, ACD into error, we can perform F-tests for the remain terms.

- B main effect is the most significant, P-value ≈ 0.02
- none of the rest is significant at 5% level

```
anova(lm(yield ~ block + B*C*(A+D), data=field))
Analysis of Variance Table

Response: yield
          Df Sum Sq Mean Sq F value Pr(>F)
block      1   2.25    2.25   0.186 0.6952
B          1 256.00  256.00  21.186 0.0193
C          1   0.25    0.25   0.021 0.8947
A          1   2.25    2.25   0.186 0.6952
D          1  20.25   20.25   1.676 0.2861
B:C        1  20.25   20.25   1.676 0.2861
B:A        1   6.25    6.25   0.517 0.5240
B:D        1  12.25   12.25   1.014 0.3882
C:A        1  81.00   81.00   6.703 0.0811
C:D        1   1.00    1.00   0.083 0.7923
B:C:A      1  16.00   16.00   1.324 0.3332
B:C:D      1 121.00  121.00  10.014 0.0507
Residuals  3  36.25   12.08
```

99% CI for B Main Effect

$$\mathrm{SE}(\widehat{\beta}_1 - \widehat{\beta}_0) = \mathrm{SE}(\bar{y}_{\bullet 1 \bullet\bullet} - \bar{y}_{\bullet 0 \bullet\bullet})$$
$$= \sqrt{\mathsf{MSE}\left(\frac{1}{8} + \frac{1}{8}\right)} = \sqrt{12.0833\left(\frac{1}{8} + \frac{1}{8}\right)} \approx 1.738$$

The t-critical value is about 5.841

```
qt(0.01/2, df=3, lower.tail=FALSE)
[1] 5.84091
```

The 99% CI for $\beta_1 - \beta_0$ is

$$-4 - 4 \pm 5.841 \times 1.738 \approx (-18.15, 2.15).$$

Adding nitrochalk might increase the yield by -18.15 to 2.15.

```
lm2 = lm(yield ~ block + B*C*(A+D), data=field)
library(emmeans)
lm2emB = emmeans(lm2, "B")
NOTE: Results may be misleading due to involvement in interactions
pairs(lm2emB, infer = c(T,T), level=0.99)
 contrast estimate   SE df lower.CL upper.CL t.ratio p.value
 0 - 1           8 1.74  3    -2.15     18.2   4.603  0.0193

Results are averaged over the levels of: block, C, A, D
Confidence level used: 0.99
```

Warning

If one analyze data using both a half-normal probability plot and by pooling terms into error, watch out that one cannot decide which terms to pool into error after looking at the half-normal plot or the ANOVA table.

- In that case, the P-values and CI's of the effects are not reliable since one tends to pool terms with small effects into error, which would lead to a small MSE and overstate the significance

Generally, only one of the two analyses can be done on one data set. They should be done using different experimental data.

- If one uses the half-normal plot to identify a set of negligible terms, one should conduct a new study, and test the significance of or construct CIs for the remaining effects using the new data.

Next Time

So far we only considered confounded 2-level factorial in *two blocks*.

With 5 or 6 or more factors, the block size (2^4, 2^5, or greater) can be too large.

Better if we could have confounded 2-level factorial in more blocks.

- 2^k designs in 4 blocks of size 2^{k-2}
- 2^k designs in 8 blocks of size 2^{k-3}