STAT 222 Lecture 20-21
Incomplete Block Designs

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Coverage

Chapter 11 Incomplete Block Designs

- Balanced Incomplete Block Designs (BIBD)
- Skip Section 11.3.2, 11.3.3, 11.4.5, 11.4.6 on Group Divisible Designs and Cyclic Designs

Incomplete Block Designs

Recall for a randomized complete block design (RCBD) of g treatments, the size k of each block has to be g (or multiples of g). Each treatment appear the same number of time(s) in a block.

In practice, the natural size of a block might not be equal to and is often smaller than the numbers of treatments ($k < g$).
We cannot include every treatment to every block.

We then have *Incomplete Block Design (IBD)*.
IBD is more difficult to analyze than complete block designs, but sometimes it's inevitable.

An Example We Must Use Incomplete Blocks

Eye irritation can be reduced with eyedrops. Three brands of eyedrops are to be compared for their ability to reduce eye irritation.

As there is a strong individual effect, subjects should be used as blocks.

If a subject can only be tested in one treatment period, the researchers can apply one brand of drop in the left eye and another brand in the right eye. The natural block size is limited to $k = 2$.

The study is force into incomplete blocks, with

$$\begin{array}{ccc} k = 2 & < & 3 = g \\ \text{(block size)} & < & \text{(number of treatments)} \end{array}$$

Example — A Marketing Psychology Experiment

- Goal: comparing 5 commercial ads: A, B, C, D, E
- Response: subjects' rating of a commercial after watching it
- A subject can watch multiple ads. A subject is a block
- Can use a RCBD of all subjects can watch all the 5 ads
- However, subjects may lose patience after watching too many ads, and they may forget the first few ads they see. Their response will be less accurate.
- To ensure the quality of the response of subjects, we may restrict the number of ads each subject watch to, say, 3. The block size is limited to $k = 3$.

Subject

1	2	3	4	5	6	7	8	9	10
A	B	E	A	C	D	B	E	D	C
B	D	A	C	A	E	C	B	E	D
C	A	B	D	E	A	D	C	B	E

Some Poor Incomplete Block Designs (1)

Block

1	2	3	4	5
A	B	C	D	E
A	B	C	D	E
A	B	C	D	E

- What's the drawback of the design above?

Some Poor Incomplete Block Designs (1)

Block

1	2	3	4	5
A	B	C	D	E
A	B	C	D	E
A	B	C	D	E

- What's the drawback of the design above?
  - block effect and treatment effects are confounded

Some Poor Incomplete Block Designs (1)

Block

1	2	3	4	5
A	B	C	D	E
A	B	C	D	E
A	B	C	D	E

- What's the drawback of the design above?
  - block effect and treatment effects are confounded
- To eliminate of block effects, better compare treatments **within a block**.

Some Poor Incomplete Block Designs (1)

Block

1	2	3	4	5
A	B	C	D	E
A	B	C	D	E
A	B	C	D	E

- What's the drawback of the design above?
  - block effect and treatment effects are confounded
- To eliminate of block effects, better compare treatments **within a block**.
- **No treatment should appear twice in any block** as they contributes nothing no **within block comparisons**.

Some Poor Incomplete Block Designs (2)

Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

can one find an unbiased estimate for

- $\alpha_A - \alpha_B$?
- $\alpha_A - \alpha_D$?
- $\alpha_A - \alpha_E$?

Some Poor Incomplete Block Designs (2) — A v.s. B

Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

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Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} \quad = \quad \mu \quad + \quad \underset{\text{(treatment)}}{\alpha_i} \quad + \quad \underset{\text{(block)}}{\beta_j} \quad + \quad \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

can one find an unbiased estimate for $\alpha_A - \alpha_B$?

Yes, $y_{A1} - y_{B1}$ is an unbiased estimate for $\alpha_A - \alpha_B$ since

$$\begin{aligned} \mathrm{E}[y_{A1} - y_{B1}] &= (\mu + \alpha_A + \beta_1) - (\mu + \alpha_B + \beta_1) \\ &= \alpha_A - \alpha_B \end{aligned}$$

Some Poor Incomplete Block Designs (2) — A v.s. D

Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

can one find an unbiased estimate for $\alpha_A - \alpha_D$?

Some Poor Incomplete Block Designs (2) — A v.s. D

Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

can one find an unbiased estimate for $\alpha_A - \alpha_D$?

Yes,

- $y_{A1} - y_{B1}$ is an unbiased estimate for $\alpha_A - \alpha_B$
- $y_{B2} - y_{D2}$ is an unbiased estimate for $\alpha_B - \alpha_D$
- Their sum $y_{A1} - y_{B1} + y_{B2} - y_{D2}$ would be an unbiased estimate for

$$(\alpha_A - \alpha_B) + (\alpha_B - \alpha_D) = \alpha_A - \alpha_D$$

Some Poor Incomplete Block Designs (2) — A v.s. E

Block

1	2	3	4
A	B	E	E
B	C	F	F
C	D	G	G

Based on the model

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

can one find an unbiased estimate for $\alpha_A - \alpha_E$?

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1	2	3	4
A	B	E	E
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Based on the model

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can one find an unbiased estimate for $\alpha_A - \alpha_E$?

Incomplete block designs must be "connected"
or not all pairwise comparisons can be estimated.

Balanced Incomplete Block Designs (BIBD)

BIBD is not balanced in the general sense that all treatment-block combinations occur equally often. Rather they are balanced in the looser sense by the criteria described below.

A *balanced incomplete block design* with

- g treatments,
- b blocks,
- k as the size of each block,
- r replications of each treatment,

is a design satisfying the following:

Incomplete:
- $k < g$.

Balanced:
- Each treatment appears at most once per block and has the same number of replicates r
- Each pair of treatments appear in a block the same number of times λ

Subject	1	2	3	4	5	6	7	8	9	10
	A	B	E	A	C	D	B	E	D	C
	B	D	A	C	A	E	C	B	E	D
	C	A	B	D	E	A	D	C	B	E

The design of the marketing psychology study is a BIBD with

$$g = \text{number of treatments} = 5$$
$$b = \text{number of blocks} = 10$$
$$k = \text{size of each block} = 3$$
$$r = \text{number replicates per treatment} = 6$$

The table below shows the blocks each treatment appears, verifying that each treatment appears $r = 6$ times.

Treatment	Block 1	2	3	4	5	6	7	8	9	10
A	√	√	√	√	√	√				
B	√	√	√				√	√	√	
C	√			√	√		√	√		√
D		√		√		√	√		√	√
E			√		√	√		√	√	√

Subject	1	2	3	4	5	6	7	8	9	10
	A	B	E	A	C	D	B	E	D	C
	B	D	A	C	A	E	C	B	E	D
	C	A	B	D	E	A	D	C	B	E

BIBD requires each pair of treatments appears in a block the same number (λ) of times. The table below verifies that, each treatment pair appears $\lambda = 3$ times for the design above.

Treatment-pair	Block 1	2	3	4	5	6	7	8	9	10
AB	$\surd$	$\surd$	$\surd$							
AC	$\surd$			$\surd$	$\surd$					
AD		$\surd$		$\surd$		$\surd$				
AE			$\surd$		$\surd$	$\surd$				
BC	$\surd$						$\surd$	$\surd$		
BD		$\surd$					$\surd$		$\surd$	
BE			$\surd$					$\surd$	$\surd$	
CD				$\surd$			$\surd$			$\surd$
CE					$\surd$			$\surd$		$\surd$
DE						$\surd$			$\surd$	$\surd$

First Balancing Condition of BIBD

The five numbers that describe a BIBD: g, b, k, r, and λ are not arbitrary.

There might not exist an allocation b blocks of k units to g treatments that is a BIBD.

- There are b blocks of size k each,
  $\Rightarrow$ total number of experimental units is $N = bk$.
- There are g treatments, each appears r times in the design
  $\Rightarrow$ total number of experimental units is $N = rg$.

Hence a BIBD must satisfy the **first balancing condition**:

$$\boxed{N = bk = rg}.$$

Second Balancing Condition of BIBD

In a BIBD, every pair of treatments must appears in a block the same number of times, say λ times.

Observe the $\boxed{\text{total number of pairings involving treatment } A}$ equals

- $\lambda(g-1)$, since A may be paired (appear in the same block) λ times with any of the other $g-1$ treatments,
- $r(k-1)$ since treatment A appears in r blocks. Within each of those blocks, there are $k-1$ pairs including A as the block size is k

The **second balancing condition**

$$\boxed{r(k-1) = \lambda(g-1)}$$

Given g treatments and b blocks of size k, one can show that a BIBD that with r replicates per treatment and each pair of treatments show in a block λ times exists if and only if

$$bk = rg \text{ and } r(k-1) = \lambda(g-1).$$

Example (Eyedrop): $g = 3$, $k = 2$.

- Is it possible to find a BIBD w/ $b = 5$ subjects (blocks)?
- Is it possible to find a BIBD w/ $b = 6$ subjects (blocks)?

Example (Marketing Psychology): $g = 5$, $k = 3$.

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- Is it possible to find a BIBD w/ $b = 5$ subjects (blocks)?
  No. $r = bk/g = 2 \times 5/3 = 10/3$ is NOT an integer.
- Is it possible to find a BIBD w/ $b = 6$ subjects (blocks)?

Example (Marketing Psychology): $g = 5$, $k = 3$.

- Is it possible to find a BIBD w/ $b = 5$ subjects (blocks)?

- Is it possible to find a BIBD w/ $b = 10$ subjects (blocks)?

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- Is it possible to find a BIBD w/ $b = 6$ subjects (blocks)?
  Yes, as $r = bk/g = 2 \times 6/3 = 4$ and $\lambda = \frac{r(k-1)}{(g-1)} = \frac{4(2-1)}{3-1} = 2$ are both integers

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Example (Marketing Psychology): $g = 5$, $k = 3$.

- Is it possible to find a BIBD w/ $b = 5$ subjects (blocks)?
  No. $r = bk/g = 3 \cdot 5/5 = 3$ is an integer, but $\lambda = \frac{r(k-1)}{(g-1)} = \frac{3(3-1)}{5-1} = 6/4$ is NOT an integer.
- Is it possible to find a BIBD w/ $b = 10$ subjects (blocks)?

Given g treatments and b blocks of size k, one can show that a BIBD that with r replicates per treatment and each pair of treatments show in a block λ times exists if and only if

$$bk = rg \text{ and } r(k-1) = \lambda(g-1).$$

Example (Eyedrop): $g = 3$, $k = 2$.

- Is it possible to find a BIBD w/ $b = 5$ subjects (blocks)?
  No. $r = bk/g = 2 \times 5/3 = 10/3$ is NOT an integer.
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- Is it possible to find a BIBD w/ $b = 10$ subjects (blocks)?
  Yes, as $r = bk/g = 10 \cdot 3/5 = 6$ and $\lambda = \frac{r(k-1)}{(g-1)} = \frac{6(3-1)}{5-1} = 3$ are both integers

Just like Latin Squares, it's not trivial to find a BIBD by oneself.

Appendix C.2 on p.609-615 of Oehlert's textbook gives a list of BIBD designs for $g \leq 9$.

- **A BIBD can be replicated to conduct a larger study.** E.g., in the marketing psychology experiment, if we have $b = 20$ subjects (blocks) instead of 10, then we can do 2 repetitions of the BIBD below with $g = 5$, $k = 3$, $b = 10$, $r = 6$, $\lambda = 3$:

A	B	E	A	C	D	B	E	D	C
B	D	A	C	A	E	C	B	E	D
C	A	B	D	E	A	D	C	B	E

- **How to Do Randomization in BIBD?** One obvious randomization is to randomize subjects to columns, then randomize the order of treatments in each block based on the above design.

Models for BIBD

$$y_{ij} = \mu + \underset{\text{(treatment)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \underset{\text{(i.i.d. } N(0,\sigma^2))}{\varepsilon_{ij}}$$

for $i = 1, \ldots, g$, and $j = 1, \ldots, b$ with

$$\sum_{i=1}^{g} \alpha_i = \sum_{j=1}^{b} \beta_j = 0.$$

- additive model (no treatment-block interaction)
- Not all y_{ij} exist because of incompleteness

Parameter Estimates for BIBD

Let

$$I_{ij} = \begin{cases} 1, & \text{if treatment } i \text{ appears in block } j, \\ 0, & \text{otherwise.} \end{cases}$$

and define

$$Q_i = y_{i\bullet} - \frac{1}{k}\sum_j I_{ij} y_{\bullet j}, \qquad Q'_j = y_{\bullet j} - \frac{1}{r}\sum_i I_{ij} y_{i\bullet}$$

the least square estimates for μ, α_i, β_j are

$$\widehat{\mu} = \frac{y_{\bullet\bullet}}{N}, \qquad \widehat{\alpha}_i = \frac{kQ_i}{\lambda g}, \qquad \widehat{\beta}_j = \frac{kQ'_j}{\lambda b}$$

Remark: Can verify that $\sum_i Q_i = 0 \quad \Rightarrow \sum_i \widehat{\alpha}_i = 0$.

You won't be asked to estimate parameters manually for a BIBD.

Example of BIBD— Problem 14.3 on. p.381 in Oehlert

The State Board of Education has adopted basic skills tests for high school graduation. One of these is a writing test. The student writing samples are graded by professional graders, and the board is taking some care to be sure that the graders are grading to the same standard. We examine grader differences with the following experiment. There are 25 graders available. We select 30 writing samples at random, each writing sample will be graded by 5 graders. Thus each grader will grade $30 \times 5/25 = 6$ samples.

Data: http://users.stat.umn.edu/~gary/book/fcdae.data/pr14.3

Questions of Interest:

- Did the 25 graders grade consistently with each other?
- How to adjust the scores if graders didn't grade consistently?
- If graders didn't grade consistently, can we identify the graders that were inconsistent with others?

Exam	Grader					Score					Exam	Grader					Score				
1	1	2	3	4	5	60	59	51	64	53	16	1	9	12	20	23	61	67	69	68	65
2	6	7	8	9	10	64	69	63	63	71	17	2	10	13	16	24	78	75	76	75	72
3	11	12	13	14	15	84	85	86	85	83	18	3	6	14	17	25	67	72	72	75	76
4	16	17	18	19	20	72	76	77	74	77	19	4	7	15	18	21	84	81	76	79	77
5	21	22	23	24	25	65	73	70	71	70	20	5	8	11	19	22	81	84	85	84	81
6	1	6	11	16	21	52	54	62	54	55	21	1	8	15	17	24	70	65	61	66	66
7	2	7	12	17	22	56	51	52	57	51	22	2	9	11	18	25	84	82	86	85	86
8	3	8	13	18	23	55	60	59	60	61	23	3	10	12	19	21	72	85	77	82	79
9	4	9	14	19	24	88	76	77	77	74	24	4	6	13	20	22	85	75	78	82	83
10	5	10	15	20	25	65	68	72	74	77	25	5	7	14	16	23	58	64	58	57	58
11	1	10	14	18	22	79	77	77	77	79	26	1	7	13	19	25	66	71	73	70	70
12	2	6	15	19	23	70	66	63	62	66	27	2	8	14	20	21	73	67	63	70	66
13	3	7	11	20	24	48	49	51	48	50	28	3	9	15	16	22	58	70	69	61	71
14	4	8	12	16	25	75	64	75	68	65	29	4	10	11	17	23	95	84	88	88	87
15	5	9	13	17	21	79	77	81	79	83	30	5	6	12	18	24	47	47	51	49	56

Here a exam is a writing sample.

- Which factor is the treatment factor? Graders or Exams?
- Which factor is the block factor? Graders or writing samples?
- Is this a BIBD?

Exam	Grader	Score	Exam	Grader	Score
1	1 2 3 4 5	60 59 51 64 53	16	1 9 12 20 23	61 67 69 68 65
2	6 7 8 9 10	64 69 63 63 71	17	2 10 13 16 24	78 75 76 75 72
3	11 12 13 14 15	84 85 86 85 83	18	3 6 14 17 25	67 72 72 75 76
4	16 17 18 19 20	72 76 77 74 77	19	4 7 15 18 21	84 81 76 79 77
5	21 22 23 24 25	65 73 70 71 70	20	5 8 11 19 22	81 84 85 84 81
6	1 6 11 16 21	52 54 62 54 55	21	1 8 15 17 24	70 65 61 66 66
7	2 7 12 17 22	56 51 52 57 51	22	2 9 11 18 25	84 82 86 85 86
8	3 8 13 18 23	55 60 59 60 61	23	3 10 12 19 21	72 85 77 82 79
9	4 9 14 19 24	88 76 77 77 74	24	4 6 13 20 22	85 75 78 82 83
10	5 10 15 20 25	65 68 72 74 77	25	5 7 14 16 23	58 64 58 57 58
11	1 10 14 18 22	79 77 77 77 79	26	1 7 13 19 25	66 71 73 70 70
12	2 6 15 19 23	70 66 63 62 66	27	2 8 14 20 21	73 67 63 70 66
13	3 7 11 20 24	48 49 51 48 50	28	3 9 15 16 22	58 70 69 61 71
14	4 8 12 16 25	75 64 75 68 65	29	4 10 11 17 23	95 84 88 88 87
15	5 9 13 17 21	79 77 81 79 83	30	5 6 12 18 24	47 47 51 49 56

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4	16	17	18	19	20	72	76	77	74	77	19	4	7	15	18	21	84	81	76	79	77
5	21	22	23	24	25	65	73	70	71	70	20	5	8	11	19	22	81	84	85	84	81
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7	2	7	12	17	22	56	51	52	57	51	22	2	9	11	18	25	84	82	86	85	86
8	3	8	13	18	23	55	60	59	60	61	23	3	10	12	19	21	72	85	77	82	79
9	4	9	14	19	24	88	76	77	77	74	24	4	6	13	20	22	85	75	78	82	83
10	5	10	15	20	25	65	68	72	74	77	25	5	7	14	16	23	58	64	58	57	58
11	1	10	14	18	22	79	77	77	77	79	26	1	7	13	19	25	66	71	73	70	70
12	2	6	15	19	23	70	66	63	62	66	27	2	8	14	20	21	73	67	63	70	66
13	3	7	11	20	24	48	49	51	48	50	28	3	9	15	16	22	58	70	69	61	71
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3	11	12	13	14	15	84	85	86	85	83	18	3	6	14	17	25	67	72	72	75	76
4	16	17	18	19	20	72	76	77	74	77	19	4	7	15	18	21	84	81	76	79	77
5	21	22	23	24	25	65	73	70	71	70	20	5	8	11	19	22	81	84	85	84	81
6	1	6	11	16	21	52	54	62	54	55	21	1	8	15	17	24	70	65	61	66	66
7	2	7	12	17	22	56	51	52	57	51	22	2	9	11	18	25	84	82	86	85	86
8	3	8	13	18	23	55	60	59	60	61	23	3	10	12	19	21	72	85	77	82	79
9	4	9	14	19	24	88	76	77	77	74	24	4	6	13	20	22	85	75	78	82	83
10	5	10	15	20	25	65	68	72	74	77	25	5	7	14	16	23	58	64	58	57	58
11	1	10	14	18	22	79	77	77	77	79	26	1	7	13	19	25	66	71	73	70	70
12	2	6	15	19	23	70	66	63	62	66	27	2	8	14	20	21	73	67	63	70	66
13	3	7	11	20	24	48	49	51	48	50	28	3	9	15	16	22	58	70	69	61	71
14	4	8	12	16	25	75	64	75	68	65	29	4	10	11	17	23	95	84	88	88	87
15	5	9	13	17	21	79	77	81	79	83	30	5	6	12	18	24	47	47	51	49	56

Here a exam is a writing sample.

- Which factor is the treatment factor? Graders or Exams?
  Graders.
- Which factor is the block factor? Graders or writing samples?
  Exams.
- Is this a BIBD?

Yes, $g = 25$, $b = 30$, $k = 5$, $r = \frac{bk}{g} = 6$, $\lambda = \frac{r(k-1)}{g-1} = 1$.

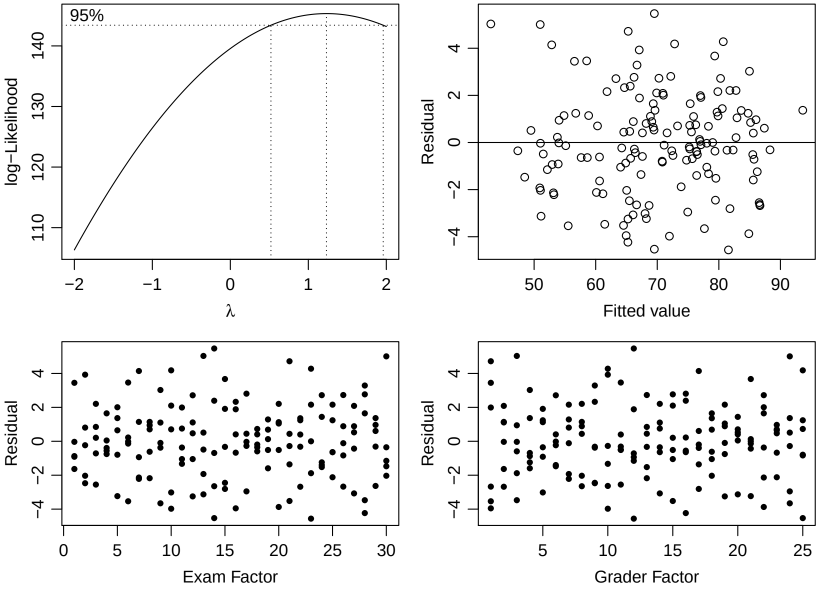
$$\underset{\text{(score)}}{y_{ij}} = \mu + \underset{\text{(grader)}}{\alpha_i} + \underset{\text{(exam)}}{\beta_j} + \varepsilon_{ij}$$

As writing samples differ in levels, we expect β_j not all equal.

If graders were consistent, they should give the same score to the same writing sample, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_{25}$

```
pr14.3 = read.table(
  "http://users.stat.umn.edu/~gary/book/fcdae.data/pr14.3", h=T)
pr14.3$EXAM = as.factor(pr14.3$exam)
pr14.3$GRADER = as.factor(pr14.3$grader)
lm1 = lm(score ~ EXAM + GRADER, data=pr14.3)
```

Always Check Model Assumptions First



How to Adjust Scores as Graders Were Inconsistent?

Based on the model $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$, the score of the ith writing sample is $\mu + \beta_j$, which is estimated by $\widehat{\mu} + \widehat{\beta}_j$.

How to get $\widehat{\beta}_j$ in R? Recall R by default estimates parameters using the baseline constraints $\alpha_1 = \beta_1 = 0$, not the zero-sum constraints $\sum_{i=1}^{g} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$.

One can use `constrasts()` and `contr.sum()` to force R using the zero-sum constraints.

```
contrasts(pr14.3$EXAM) = contr.sum(30)
contrasts(pr14.3$GRADER) = contr.sum(25)
lm1 = lm(score ~ EXAM + GRADER, data=pr14.3)
```

```
lm1$coef
(Intercept)        EXAM1        EXAM2    .... (omitted)
     69.960      -12.568       -3.368
     EXAM28       EXAM29      GRADER1    .... (omotted)    GRADER24
     -2.128       16.192       -0.840                        0.160
```

Why is there no estimate for exam #30, nor for grader #25?

```r
muhat = lm1$coef[1]
betahat = vector("numeric",length=30)
betahat[1:29] = lm1$coef[2:30]
betahat[30] = -sum(betahat[1:29])
adjustedscore = muhat + betahat; adjustedscore
 [1] 57.39 66.59 84.39 75.15 69.47 56.38 51.62 60.42 77.50 71.50
[11] 77.85 65.65 49.33 68.21 80.57 65.79 74.79 73.95 78.11 83.35
[21] 66.12 83.44 80.24 78.76 60.24 69.51 67.67 67.83 86.15 50.83
names(adjustedscore) = 1:30
adjustedscore
    1     2     3     4     5     6     7     8     9    10
57.39 66.59 84.39 75.15 69.47 56.38 51.62 60.42 77.50 71.50
   11    12    13    14    15    16    17    18    19    20
77.85 65.65 49.33 68.21 80.57 65.79 74.79 73.95 78.11 83.35
   21    22    23    24    25    26    27    28    29    30
66.12 83.44 80.24 78.76 60.24 69.51 67.67 67.83 86.15 50.83
```

Compare adjusted scores with unadjusted scores (average of the 5 raw scores per exam).

```
library(mosaic)
unadjustedscore = mean(score ~ EXAM, data=pr14.3)
unadjustedscore
   1    2    3    4    5    6    7    8    9   10   11   12   13 
57.4 66.0 84.6 75.2 69.8 55.4 53.4 59.0 78.4 71.2 77.8 65.4 49.2 
  14   15   16   17   18   19   20   21   22   23   24   25   26 
69.4 79.8 66.0 75.2 72.4 79.4 83.0 65.6 84.6 79.0 80.6 59.0 70.0 
  27   28   29   30 
67.8 65.8 88.4 50.0 
```

Difference of unadjusted and adjusted scores:

```
sort(unadjustedscore - adjustedscore)
    28     18      8     23     25      6     30     15      2 
-2.032 -1.552 -1.416 -1.240 -1.240 -0.976 -0.832 -0.768 -0.592 
    21     20     10     12     13     11      1      4     27 
-0.520 -0.352 -0.296 -0.248 -0.128 -0.048  0.008  0.048  0.128 
     3     16      5     17     26      9     22     14     19 
 0.208  0.208  0.328  0.408  0.488  0.904  1.160  1.192  1.288 
     7     24     29 
 1.784  1.840  2.248 
```

ANOVA Table for BIBD

```
anova(lm(score ~ GRADER + EXAM, data=pr14.3))
Analysis of Variance Table

Response: score
          Df  Sum Sq Mean Sq F value    Pr(>F)
GRADER    24  4073.1  169.71  23.659 < 2.2e-16
EXAM      29 13342.0  460.07  64.138 < 2.2e-16
Residuals 96   688.6    7.17
anova(lm(score ~ EXAM + GRADER, data=pr14.3))
Analysis of Variance Table

Response: score
          Df  Sum Sq Mean Sq F value        Pr(>F)
EXAM      29 16609.0  572.72 79.8424     < 2.2e-16
GRADER    24   806.2   33.59  4.6828 0.00000002694
Residuals 96   688.6    7.17
```

The two ANOVA tables have identical SSE
but *different SS* for `EXAM` and `GRADER`. Why?

- As a BIBD doesn't include all treatment-block combination, it does NOT have a *balanced* factorial structure of treatment $\times$ block.
- For unbalanced factorial data, there are 3 types of sum of squares
  - the `anova()` command gives the *Type I sum of squares*
  - What's a Type I Sum of Square?

Digression: Sum of Squares for Unbalanced Factorial Data

What Happens If Factorial Data Become Unbalanced

- no simple formulae for parameter estimates and SS.
- the parameter estimates and SS of a term will depend on the presence of other terms in the model, e.g., the estimates for α_i's might be different in the following 3 models

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}$$
$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$$
$$y_{ijk} = \mu + \alpha_i + \varepsilon_{ijk}$$

- need to rely on statistical software for computation
- there are 3 variations of SS

Notation for Models

In the following, we denote various models by listing the included effect. For example,

- $(1, A, B, AB)$ denotes the model $y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}$
- $(1, A, B)$ denotes the model $y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$
- $(1, A, B, C, AB, AC)$ denotes the model

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \alpha\beta_{ij} + \alpha\gamma_{ik} + \varepsilon_{ijkl}$$

Here the "**1**" stands for the grand mean μ.

Notation for Models

In the following, we denote various models by listing the included effect. For example,

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- $(1, A, B, C, AB, AC)$ denotes the model

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \alpha\beta_{ij} + \alpha\gamma_{ik} + \varepsilon_{ijkl}$$

Here the "$\mathbf{1}$" stands for the grand mean μ.

In the following SSE(model) denotes the SSE of that model, e.g., SSE$(1, A, B, AB)$ means the SSE of the model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}.$$

For unbalanced data, there is no simple formula to compute the SSE. One must write the model as a regression model and use statistical software to compute the SSE.

Adjusted Sum of Squares (1)

The **adjusted sum of squares** for main effects B adjusted for A is defined as

$$SS(B|1, A) = \text{SSE}(1, A) - \text{SSE}(1, A, B).$$

- $SS(B|1, A) \geq 0$ since the model $(1, A)$ is *included* (*nested*) in the model $(1, A, B)$ and hence the latter always has a smaller SSE
- $SS(B|1, A)$ is the reduction in SSE after B is included in the model
- $SS(B|1, A)$ describes the effect of B adjusted for A since with consider two models that A is present in both and the two models only differ by B

Adjusted Sum of Squares (2)

Likewise, the **adjusted sum of squares** for main effects B adjusted for A, C, and AC is

$$SS(B|1, A, C, AC) = \text{SSE}(1, A, C, AC) - \text{SSE}(1, A, B, C, AC).$$

Adjusted Sum of Squares (2)

Likewise, the **adjusted sum of squares** for main effects B adjusted for A, C, and AC is

$$SS(B|1, A, C, AC) = \mathrm{SSE}(1, A, C, AC) - \mathrm{SSE}(1, A, B, C, AC).$$

In general, the **adjusted sum of squares** for a term adjusted for some other terms is

$$\begin{aligned} & SS(\text{a term}|\texttt{some other terms}) \\ = & \,\mathrm{SSE}(\texttt{some other terms}) - \mathrm{SSE}(\text{a term}, \texttt{some other terms}) \end{aligned}$$

Adjusted Sum of Squares (2)

Likewise, the **adjusted sum of squares** for main effects B adjusted for A, C, and AC is

$$SS(B|1, A, C, AC) = \text{SSE}(1, A, C, AC) - \text{SSE}(1, A, B, C, AC).$$

In general, the **adjusted sum of squares** for a term adjusted for some other terms is

$$\begin{aligned} &SS(\text{a term}|\texttt{some other terms}) \\ =& \text{SSE}(\texttt{some other terms}) - \text{SSE}(\text{a term}, \texttt{some other terms}) \end{aligned}$$

For balanced data, adjusted SS = unadjusted SS

$$SS(A|1, B) = SS(A|1, B, C) = SS(A|1, B, C, BC) = SS(A|1).$$

Type I Sum of Squares

For a specified model, the *Type I Sum of Squares* (aka. *Sequential Sum of Squares*) for any term is adjusted for those terms that precede it in the model.

- E.g, the Type I SS's for the model (1, A, B, AB, C) are

Source	d.f.	Type I SS
A	$a-1$	$SS(A\|1)$
B	$b-1$	$SS(B\|1, A)$
AB	$(a-1)(b-1)$	$SS(AB\|1, A, B)$
C	$c-1$	$SS(C\|1, A, B, AB)$

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AB	$(a-1)(b-1)$	$SS(AB\|1,A,B)$
C	$c-1$	$SS(C\|1,A,B,AB)$

Type I SS's depend on how the terms are **ordered** in a model:

- E.g, if the terms in the model (1, A, B, AB, C) is reshuffled as (1, C, A, B, AB), then the Type I SS's become

Source	d.f.	Type I SS
C	$c-1$	$SS(C\|1)$
A	$a-1$	$SS(A\|1,C)$
B	$b-1$	$SS(B\|1,A,C)$
AB	$(a-1)(b-1)$	$SS(AB\|1,A,B,C)$

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	~~$\text{SSE}(1, A, B, AB, C)$~~
Error	~~$\text{SSE}(1, A, B, AB, C)$~~				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\cancel{\text{SSE}(1, A, B, AB)}$
C	$SS(C\|1, A, B, AB)$	$=$	$\cancel{\text{SSE}(1, A, B, AB)}$	$-$	$\cancel{\text{SSE}(1, A, B, AB, C)}$
Error			$\cancel{\text{SSE}(1, A, B, AB, C)}$		
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, A)$
B	$SS(B\|1, A)$	$=$	$\text{SSE}(1, A)$	$-$	$\text{SSE}(1, A, B)$
AB	$SS(AB\|1, A, B)$	$=$	$\text{SSE}(1, A, B)$	$-$	$\text{SSE}(1, A, B, AB)$
C	$SS(C\|1, A, B, AB)$	$=$	$\text{SSE}(1, A, B, AB)$	$-$	$\text{SSE}(1, A, B, AB, C)$
Error	$\text{SSE}(1, A, B, AB, C)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (1)

Source	Type I SS				
A	$SS(A \mid 1)$	$=$	$\text{SSE}(1)$	$-$	$\cancel{\text{SSE}(1, A)}$
B	$SS(B \mid 1, A)$	$=$	$\cancel{\text{SSE}(1, A)}$	$-$	$\cancel{\text{SSE}(1, A, B)}$
AB	$SS(AB \mid 1, A, B)$	$=$	$\cancel{\text{SSE}(1, A, B)}$	$-$	$\cancel{\text{SSE}(1, A, B, AB)}$
C	$SS(C \mid 1, A, B, AB)$	$=$	$\cancel{\text{SSE}(1, A, B, AB)}$	$-$	$\cancel{\text{SSE}(1, A, B, AB, C)}$
Error	$\cancel{\text{SSE}(1, A, B, AB, C)}$				
Sum	$\text{SSE}(1) = \text{SST}$				

SSE(1) is the SSE for the model $y_{ijk\ell} = \mu + \varepsilon_{ijk\ell}$, of which the optimal (least square) estimate for μ is the overall mean $\overline{y}_{\bullet\bullet\bullet\bullet}$. Hence,

$$\text{SSE}(1) = \sum_{ijk\ell}(y_{ijk\ell} - \overline{y}_{\bullet\bullet\bullet\bullet})^2 = \text{SST}.$$

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
- $\text{SSE}(1, A, B, AB, C) = \text{SSE}(1, C, A, B, AB)$ is not affected by the order of terms;
- the Type I SS's and the SSE always add up to SST.

Source	Type I SS				
C	$SS(C\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, C)$
A	$SS(A\|1, C)$	$=$	$\text{SSE}(1, C)$	$-$	$\text{SSE}(1, C, A)$
B	$SS(B\|1, C, A)$	$=$	$\text{SSE}(1, C, A)$	$-$	$\text{SSE}(1, C, A, B)$
AB	$SS(AB\|1, A, B, C)$	$=$	$\text{SSE}(1, C, A, B)$	$-$	$\text{SSE}(1, A, B, C, AB)$
Error	$\text{SSE}(1, C, A, B, AB)$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
- $\text{SSE}(1, A, B, AB, C) = \text{SSE}(1, C, A, B, AB)$ is not affected by the order of terms;
- the Type I SS's and the SSE always add up to SST.

Source	Type I SS
C	$SS(C\|1) = \text{SSE}(1) - \text{SSE}(1, C)$
A	$SS(A\|1, C) = \text{SSE}(1, C) - \text{SSE}(1, C, A)$
B	$SS(B\|1, C, A) = \text{SSE}(1, C, A) - \text{SSE}(1, C, A, B)$
AB	$SS(AB\|1, A, B, C) = \text{SSE}(1, C, A, B) - \text{SSE}(1, A, B, C, AB)$
Error	$\text{SSE}(1, C, A, B, AB)$
Sum	

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If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

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C	$SS(C\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, C)$
A	$SS(A\|1, C)$	$=$	$\text{SSE}(1, C)$	$-$	$\text{SSE}(1, C, A)$
B	$SS(B\|1, C, A)$	$=$	$\text{SSE}(1, C, A)$	$-$	$\text{SSE}(1, C, A, B)$
AB	$SS(AB\|1, A, B, C)$	$=$	$\text{SSE}(1, C, A, B)$	$-$	$\cancel{\text{SSE}(1, A, B, C, AB)}$
Error	$\cancel{\text{SSE}(1, C, A, B, AB)}$				
Sum					

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
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Source	Type I SS				
C	$SS(C\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, C)$
A	$SS(A\|1, C)$	$=$	$\text{SSE}(1, C)$	$-$	$\text{SSE}(1, C, A)$
B	$SS(B\|1, C, A)$	$=$	$\text{SSE}(1, C, A)$	$-$	~~$\text{SSE}(1, C, A, B)$~~
AB	$SS(AB\|1, A, B, C)$	$=$	~~$\text{SSE}(1, C, A, B)$~~	$-$	~~$\text{SSE}(1, A, B, C, AB)$~~
Error	~~$\text{SSE}(1, C, A, B, AB)$~~				
Sum					

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
- $\text{SSE}(1, A, B, AB, C) = \text{SSE}(1, C, A, B, AB)$ is not affected by the order of terms;
- the Type I SS's and the SSE always add up to SST.

Source	Type I SS				
C	$SS(C\|1)$	$=$	$\text{SSE}(1)$	$-$	$\text{SSE}(1, C)$
A	$SS(A\|1, C)$	$=$	$\text{SSE}(1, C)$	$-$	~~$\text{SSE}(1, C, A)$~~
B	$SS(B\|1, C, A)$	$=$	~~$\text{SSE}(1, C, A)$~~	$-$	~~$\text{SSE}(1, C, A, B)$~~
AB	$SS(AB\|1, A, B, C)$	$=$	~~$\text{SSE}(1, C, A, B)$~~	$-$	~~$\text{SSE}(1, A, B, C, AB)$~~
Error	~~$\text{SSE}(1, C, A, B, AB)$~~				
Sum					

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
- $\text{SSE}(1, A, B, AB, C) = \text{SSE}(1, C, A, B, AB)$ is not affected by the order of terms;
- the Type I SS's and the SSE always add up to SST.

Source	Type I SS				
C	$SS(C\|1)$	$=$	$\text{SSE}(1)$	$-$	~~$\text{SSE}(1, C)$~~
A	$SS(A\|1, C)$	$=$	~~$\text{SSE}(1, C)$~~	$-$	~~$\text{SSE}(1, C, A)$~~
B	$SS(B\|1, C, A)$	$=$	~~$\text{SSE}(1, C, A)$~~	$-$	~~$\text{SSE}(1, C, A, B)$~~
AB	$SS(AB\|1, A, B, C)$	$=$	~~$\text{SSE}(1, C, A, B)$~~	$-$	~~$\text{SSE}(1, A, B, C, AB)$~~
Error	~~$\text{SSE}(1, C, A, B, AB)$~~				
Sum					

Type I Sum of Squares and SSE Add Up to SST (2)

If the order of terms in the model $(1, A, B, AB, C)$ is changed to $(1, C, A, B, AB)$,

- the Type I SS's are changed;
- $\text{SSE}(1, A, B, AB, C) = \text{SSE}(1, C, A, B, AB)$ is not affected by the order of terms;
- the Type I SS's and the SSE always add up to SST.

Source	Type I SS				
C	$SS(C\|1)$	$=$	SSE(1)	$-$	~~SSE(1, C)~~
A	$SS(A\|1, C)$	$=$	~~SSE(1, C)~~	$-$	~~SSE(1, C, A)~~
B	$SS(B\|1, C, A)$	$=$	~~SSE(1, C, A)~~	$-$	~~SSE(1, C, A, B)~~
AB	$SS(AB\|1, A, B, C)$	$=$	~~SSE(1, C, A, B)~~	$-$	~~SSE(1, A, B, C, AB)~~
Error	~~SSE(1, C, A, B, AB)~~				
Sum	$\text{SSE}(1) = \text{SST}$				

Example: Popcorn Microwave Data Revisit

$3 \times 2 \times 3$ factorial design with 2 replicates.

Brand (i)	Power (j)	Time (k)		
		1 (4 min)	2 (4.5 min)	3 (5 min)
1	1 (500 W)	73.8, 65.5	70.3, 91.0	72.7, 81.9
1	2 (625 W)	70.8, 75.3	78.7, 88.7	74.1, 72.1
2	1 (500 W)	73.7, 65.8	93.4, 76.3	45.3, 47.6
2	2 (625 W)	79.3, 86.5	92.2, 84.7	66.3, 45.7
3	1 (500 W)	62.5, 65.0	50.1, 81.5	51.4, 67.7
3	2 (625 W)	82.1, 74.5	71.5, 80.0	64.0, 77.0

```
popcorn = read.table(
          "http://www.stat.uchicago.edu/~yibi/s222/popcorn.txt", h=T)
popcorn$brand = as.factor(popcorn$brand)
popcorn$power = as.factor(popcorn$power)
popcorn$time = as.factor(popcorn$time)
```

For balanced data, SS's are not affected by the order of the terms in the model

```
anova(lm(y ~ brand*time+power, data=popcorn))
Analysis of Variance Table

Response: y
           Df  Sum Sq Mean Sq F value    Pr(>F)
brand       2  331.10  165.55  2.3031 0.1199906
time        2 1554.58  777.29 10.8133 0.0003825
power       1  455.11  455.11  6.3313 0.0183703
brand:time  4 1433.86  358.46  4.9868 0.0040523
Residuals  26 1868.95   71.88
anova(lm(y ~ power+brand*time, data=popcorn))
Analysis of Variance Table

Response: y
           Df  Sum Sq Mean Sq F value    Pr(>F)
power       1  455.11  455.11  6.3313 0.0183703
brand       2  331.10  165.55  2.3031 0.1199906
time        2 1554.58  777.29 10.8133 0.0003825
brand:time  4 1433.86  358.46  4.9868 0.0040523
Residuals  26 1868.95   71.88
```

If the first observation (73.8) is removed `popcorn[-1,]` is removed, the data become unbalanced.

```
anova(lm(y ~ brand*time+power, data=popcorn[-1,]))
Analysis of Variance Table

Response: y
           Df  Sum Sq Mean Sq F value    Pr(>F)
brand       2  334.95  167.48  2.3017 0.1209104
time        2 1559.57  779.79 10.7172 0.0004353
power       1  443.81  443.81  6.0996 0.0206998
brand:time  4 1483.60  370.90  5.0975 0.0038263
Residuals  25 1819.02   72.76
anova(lm(y ~ power+brand*time, data=popcorn[-1,]))
Analysis of Variance Table

Response: y
           Df  Sum Sq Mean Sq F value    Pr(>F)
power       1  480.66  480.66  6.6061 0.0165065
brand       2  304.29  152.15  2.0911 0.1446414
time        2 1553.38  776.69 10.6746 0.0004454
brand:time  4 1483.60  370.90  5.0975 0.0038263
Residuals  25 1819.02   72.76
```

Type I ANOVA table

The Type I ANOVA table for unbalanced data are identical to the ANOVA table for balanced data in every aspect except the SS's are replaced by the Type I SS.

Source	d.f.	Type I SS	MS	F-value
A	$a-1$	$SS(A\|1)$	SS_A/df_A	MS_A/MSE
B	$b-1$	$SS(B\|1,A)$	SS_B/df_B	MS_B/MSE
C	$c-1$	$SS(C\|1,A,B)$	SS_C/df_C	MS_C/MSE
AB	$(a-1)(b-1)$	$SS(AB\|1,A,B,C)$	SS_{AB}/df_{AB}	$\text{MS}_{AB}/\text{MSE}$
AC	$(a-1)(c-1)$	$SS(AC\|1,A,B,C,AB)$	SS_{AC}/df_{AC}	$\text{MS}_{AC}/\text{MSE}$
BC	$(a-1)(c-1)$	$SS(BC\|1,A,B,C,AB,AC)$	SS_{BC}/df_{BC}	$\text{MS}_{BC}/\text{MSE}$
ABC	$(a-1)(b-1)(c-1)$	$SS(ABC\|1,A,B,C,AB,AC,BC)$	SS_{ABC}/df_{ABC}	$\text{MS}_{ABC}/\text{MSE}$
Error	$N-abc$	SSE	SSE/df_E	
Total	$N-1$	SST		

Type I SS's and the SSE always add up to SST.

Why Type I SS's Are Not Ideal?

Look the 3 F-statistic for the 3 main effects in the previous page.

- The F-statistic for A is unadjusted
- The F-statistic for B is adjusted with A
- The F-statistic for C is adjusted with both A and B

When considering whether a term, say A, is needed in a model, one should look at the *net effect* of A after adjusting for the effect of other terms.

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$$1, B, C, BC.$$

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- Why not adjusting for AB, AC and ABC?
- Thus, a more sensible adjusted SS for A is $SS(A|1, B, C, BC)$.

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When considering whether a term, say A, is needed in a model, one should look at the *net effect* of A after adjusting for the effect of other terms.

- What are the terms that should be accounted for before considering A?
$$1, B, C, BC.$$
- Why not adjusting for AB, AC and ABC?
- Thus, a more sensible adjusted SS for A is $SS(A|1, B, C, BC)$.
- Such adjusted SS's are called the *Type II Sum of Squares*.

Type II Sum of Squares

The Type II SS_U of an effect U (U can be a main effect or an interaction) is computed as follows:

- take the biggest hierarchical model without effect U, and then compare it to the model with U added.

Here "largest hierarchical model" means all the effects that don't include term U. E.g., for the model (1, A, B, C, AB, AC, BC, ABC),

- the Type II SS for AB is $SS(AB|1, A, B, C, AC, BC)$
- the Type II SS for C is $SS(C|1, A, B, AB)$ but not $SS(C|1, A)$ or $SS(C|1, A, AB)$

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Here "largest hierarchical model" means all the effects that don't include term U. E.g., for the model (1, A, B, C, AB, AC, BC, ABC),

- the Type II SS for AB is $SS(AB|1, A, B, C, AC, BC)$
- the Type II SS for C is $SS(C|1, A, B, AB)$ but not $SS(C|1, A)$ or $SS(C|1, A, AB)$

Unlike Type I SS, Type II SS does NOT depend on the *order* of terms in a model.

Type II ANOVA table for 3-Way Data

Source	d.f.	Type II SS	MS	F-value
A	$a-1$	$\text{SS}(A\|1, B, C, BC)$	SS_A/df_A	MS_A/MSE
B	$b-1$	$\text{SS}(B\|1, A, C, AC)$	SS_B/df_B	MS_B/MSE
C	$c-1$	$\text{SS}(C\|1, A, B, AB)$	SS_C/df_C	MS_C/MSE
AB	$(a-1)(b-1)$	$\text{SS}(AB\|1, A, B, C, AC, BC)$	SS_{AB}/df_{AB}	$\text{MS}_{AB}/\text{MSE}$
AC	$(a-1)(c-1)$	$\text{SS}(AC\|1, A, B, C, AB, BC)$	SS_{AC}/df_{AC}	$\text{MS}_{AC}/\text{MSE}$
BC	$(a-1)(c-1)$	$\text{SS}(BC\|1, A, B, C, AB, AC)$	SS_{BC}/df_{BC}	$\text{MS}_{BC}/\text{MSE}$
ABC	$(a-1)(b-1)(c-1)$	$\text{SS}(ABC\|1, A, B, C, AB, AC, BC)$	SS_{ABC}/df_{ABC}	$\text{MS}_{ABC}/\text{MSE}$
Error	$N-abc$	SSE	SSE/df_E	

Type II SS of terms in a model will NOT sum to SST

Computing Type II ANOVA Table in R

The build-in function `anova()` in R gives Type I SS's only. To get the Type II SS's, first load the library `car` (which is the short for "Companion to Applied Regression"), and then use the function `Anova()` as follows.

```
library(car)
Anova(yourmodel, type=2)
```

Note the first letter `A` in `Anova()` is a capital letter A.

Type II ANOVA table:

```
library(car)
lm2 = lm(y ~ brand*power*time, data=popcorn[-1,])
Anova(lm2, type=2)
Anova Table (Type II tests)

Response: y
                  Sum Sq Df F value   Pr(>F)
brand             292.02  2  1.6082 0.229256
power             497.05  1  5.4747 0.031753
time             1559.33  2  8.5876 0.002644
brand:power       141.08  2  0.7770 0.475450
brand:time       1464.49  4  4.0326 0.017689
power:time         68.18  2  0.3755 0.692505
brand:power:time   49.33  4  0.1358 0.966830
Residuals        1543.43 17
```

```
lm2 = lm(y ~ brand*power*time, data=popcorn[-1,])
anova(lm2)
Analysis of Variance Table

Response: y
                 Df  Sum Sq Mean Sq F value   Pr(>F)
brand             2  334.95  167.48  1.8447 0.188353
power             1  450.00  450.00  4.9565 0.039804
time              2 1553.38  776.69  8.5548 0.002688
brand:power       2  205.72  102.86  1.1330 0.345227
brand:time        4 1435.95  358.99  3.9540 0.019026
power:time        2   68.18   34.09  0.3755 0.692505
brand:power:time  4   49.33   12.33  0.1358 0.966830
Residuals        17 1543.43   90.79
```

Note the Type I ANOVA table given by build-in `anova()` command is different from the Type II table given by the `Anova()` in the `car` library.

Back to BIBD

ANOVA for BIBD (Type I Sum of Squares!)

Source	d.f.	SS	MS	F-value
Block	$b-1$	SS_{block}	MS_{block}	(MS_{block}/MSE)
Treatment	$g-1$	SS_{trt}	MS_{trt}	MS_{trt}/MSE
Error	$N-g-b+1$	SSE	MSE	
Total	$N-1$	SS_{total}		

Let $I_{ij} = \begin{cases} 1, & \text{if treatment } i \text{ appears in block } j, \\ 0, & \text{otherwise.} \end{cases}$

Then $SS_{total} = \sum_{i=1}^{g} \sum_{j=1}^{b} I_{ij}(y_{ij} - \overline{y}_{\bullet\bullet})^2$

$$SS_{block} = k \sum_{j=1}^{b} (\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet})^2 \qquad \text{(unadjusted, Type I)}$$

$$SS_{trt} = \frac{k}{\lambda g} \sum_i Q_i^2 = \frac{\lambda g}{k} \sum_i \widehat{\alpha}_i^2 \quad \text{(adjusted for block, Type I \& II)}$$

$$SSE = SS_{total} - SS_{block} - SS_{trt}$$

ANOVA for BIBD (Type I Sum of Squares!)

Source	d.f.	SS	MS	F-value
Block	$b-1$	SS_{block}	MS_{block}	$(\text{MS}_{block}/\text{MSE})$
Treatment	$g-1$	SS_{trt}	MS_{trt}	$\text{MS}_{trt}/\text{MSE}$
Error	$N-g-b+1$	SSE	MSE	
Total	$N-1$	SS_{total}		

Let $\quad I_{ij} = \begin{cases} 1, & \text{if treatment } i \text{ appears in block } j, \\ 0, & \text{otherwise.} \end{cases}$

Then $SS_{total} = \sum_{i=1}^{g} \sum_{j=1}^{b} I_{ij}(y_{ij} - \overline{y}_{\bullet\bullet})^2$

$$SS_{block} = k \sum_{j=1}^{b} (\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet})^2 \qquad \text{(unadjusted, Type I)}$$

$$SS_{trt} = \frac{k}{\lambda g} \sum_i Q_i^2 = \frac{\lambda g}{k} \sum_i \widehat{\alpha}_i^2 \quad \text{(adjusted for block, Type I \& II)}$$

$$SSE = SS_{total} - SS_{block} - SS_{trt}$$

For incomplete block designs, **always place *Block ahead of Treatment* in the ANOVA table. The SS_{trt} will then be adjusted for Block and hence is Type II.**

```
anova(lm(score ~ GRADER + EXAM, data=pr14.3))
Analysis of Variance Table

Response: score
          Df   Sum Sq Mean Sq F value     Pr(>F)
GRADER    24  4073.09 169.712 23.6593 < 2.22e-16
EXAM      29 13342.04 460.070 64.1377 < 2.22e-16
Residuals 96   688.62   7.173
anova(lm(score ~ EXAM + GRADER, data=pr14.3))
Analysis of Variance Table

Response: score
          Df   Sum Sq Mean Sq  F value         Pr(>F)
EXAM      29 16608.96 572.723 79.84239    < 2.22e-16
GRADER    24   806.18  33.591  4.68282 0.00000002694
Residuals 96   688.62   7.173
```

Which ANOVA table should we look at to determine the significance of treatment (GRADER)?

Pairwise Comparisons

Estimate of $\alpha_{i_1} - \alpha_{i_2}$ is

$$\widehat{\alpha}_{i_1} - \widehat{\alpha}_{i_2} = \frac{k}{\lambda g}(Q_{i_1} - Q_{i_2})$$

- $\text{SE}(\widehat{\alpha}_{i_1} - \widehat{\alpha}_{i_2}) = \sqrt{\text{MSE}\left(\frac{2k}{\lambda g}\right)}$
- t-statistic $= \dfrac{\widehat{\alpha}_{i_1} - \widehat{\alpha}_{i_2}}{\text{SE}}$ with df $=$ df of MSE
- Tukey's HSD controlling FWER at α is

$$\text{HSD} = \frac{q_\alpha(g, \text{df of MSE})}{\sqrt{2}} \times \text{SE}.$$

How to Identify Inconsistent Graders?

We can do pairwise comparisons for the grader effects $\alpha_{i_1} - \alpha_{i_2}$ using the t-statistic $= \dfrac{\widehat{\alpha}_{i_1} - \widehat{\alpha}_{i_2}}{\text{SE}}$ where

$$SE = \sqrt{\text{MSE}\left(\frac{2k}{\lambda g}\right)} = \sqrt{7.173\left(\frac{2 \times 5}{1 \times 25}\right)} \approx 1.6939$$

with df $=$ (df of MSE) $= 96$.

Tukey's critical value for FWER $= 0.05$ is

```
qtukey(0.95, 25, df = 96)/sqrt(2)
[1] 3.768
```

$$\text{Tukey's HSD} = \frac{q_{0.05}(25, 96)}{\sqrt{2}} SE = 3.7676 \times 1.6939 \approx 6.382.$$

We have obtained $\widehat{\alpha}_1, \widehat{\alpha}_2, \ldots, \widehat{\alpha}_{24}$ in R on page 21.

```
lm1$coef[31:54]
 GRADER1  GRADER2  GRADER3  GRADER4  GRADER5  GRADER6  GRADER7
   -0.84     3.24    -6.36     7.48    -3.48    -2.36     1.60
 GRADER8  GRADER9 GRADER10 GRADER11 GRADER12 GRADER13 GRADER14
   -1.56    -1.12     0.48     2.16     1.32     0.76    -1.60
GRADER15 GRADER16 GRADER17 GRADER18 GRADER19 GRADER20 GRADER21
   -1.60    -2.60     1.24     0.20    -0.40     1.80    -1.24
GRADER22 GRADER23 GRADER24
    1.52    -0.12     0.16
```

The last one can be computed as $\widehat{\alpha}_{25} = -\sum_{i=1}^{24} \widehat{\alpha}_i = 1.32$ as $\sum_{i=1}^{25} \widehat{\alpha}_i = 0$.

```
alphahat25 = -sum(lm1$coef[31:54]); alphahat25
[1] 1.32
names(alphahat25) = "GRADER25"
alphahat = c(lm1$coef[31:54], alphahat25)
```

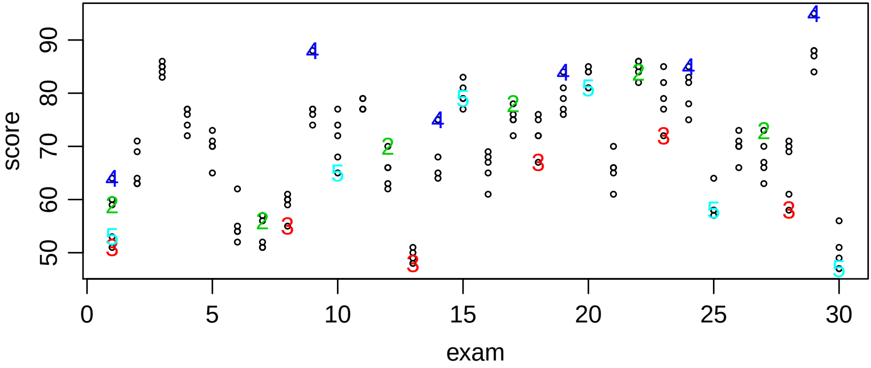
```
sort(alphahat)
 GRADER3  GRADER5 GRADER16  GRADER6 GRADER15 GRADER14  GRADER8
   -6.36    -3.48    -2.60    -2.36    -1.60    -1.60    -1.56
GRADER21  GRADER9  GRADER1 GRADER19 GRADER23 GRADER24 GRADER18
   -1.24    -1.12    -0.84    -0.40    -0.12     0.16     0.20
GRADER10 GRADER13 GRADER17 GRADER25 GRADER12 GRADER22  GRADER7
    0.48     0.76     1.24     1.32     1.32     1.52     1.60
GRADER20 GRADER11  GRADER2  GRADER4
    1.80     2.16     3.24     7.48
```

Underline Diagram for pairwise comparison between graders:
(at FWER = 5%, Tukey's HSD = 6.38)

3 5 16 6 15 14 8 21 9 1 19 23 24 18 10 13 17 25 12 22 7 20 11 2 4

After Tukey's adjustment, only Grader #3 and # 4 are significantly inconsistent with most other graders.

Grader #2 and #5 were consistent with all the rest except #3 and #4.



- Grader #3 always gave the lowest score among the 5 graders grading the same exam
- Grader #4 always gave scores that substantially higher than the scores given by the other graders for the same exam.
- Grader #2 tends to give higher scores, Grader #5 tended to give lower scores, but not as much as Grader #3 and #4.

Tukey's HSD in emmeans

The `emmeans` library also works for incomplete block designs.

```
library(emmeans)
lm1 = lm(score ~ EXAM + GRADER, data=pr14.3)
lm1em = emmeans(lm1, "GRADER")
summary(pairs(lm1em, infer=c(T,T), level=0.95, adjust="tukey"))
```

Output on the next page.

Observe the CI's all equals to their respective estimate $\pm$ HSD. e.g., the CI for Grade #1- Grade #2 is

$$-4.08 \pm 6.382 = (2.302, -10.462).$$

```
 contrast estimate   SE df lower.CL upper.CL t.ratio p.value
 1 - 2       -4.08 1.69 96  -10.462    2.302  -2.409  0.7545
 1 - 3        5.52 1.69 96   -0.862   11.902   3.259  0.1919
 1 - 4       -8.32 1.69 96  -14.702   -1.938  -4.912  0.0009
 1 - 5        2.64 1.69 96   -3.742    9.022   1.559  0.9973
 1 - 6        1.52 1.69 96   -4.862    7.902   0.897  1.0000
 1 - 7       -2.44 1.69 96   -8.822    3.942  -1.440  0.9991
 1 - 8        0.72 1.69 96   -5.662    7.102   0.425  1.0000
 1 - 9        0.28 1.69 96   -6.102    6.662   0.165  1.0000
 1 - 10      -1.32 1.69 96   -7.702    5.062  -0.779  1.0000
 1 - 11      -3.00 1.69 96   -9.382    3.382  -1.771  0.9857
 1 - 12      -2.16 1.69 96   -8.542    4.222  -1.275  0.9999
 1 - 13      -1.60 1.69 96   -7.982    4.782  -0.945  1.0000
 1 - 14       0.76 1.69 96   -5.622    7.142   0.449  1.0000
 1 - 15       0.76 1.69 96   -5.622    7.142   0.449  1.0000
 1 - 16       1.76 1.69 96   -4.622    8.142   1.039  1.0000
 [ reached getOption("max.print") -- omitted 285 rows ]

Results are averaged over the levels of: EXAM
Confidence level used: 0.95
Conf-level adjustment: tukey method for comparing a family of 25 estima
P value adjustment: tukey method for comparing a family of 25 estimates
```

```
out = summary(pairs(lm1em, infer=c(F,T), level=0.95, adjust="tukey"))
subset(out, out$p.value < 0.05)
   contrast estimate     SE df t.ratio          p.value
3     1 - 4    -8.32 1.6939 96 -4.9118 0.00093757984520
25    2 - 3     9.60 1.6939 96  5.6674 0.00004203552076
27    2 - 5     6.72 1.6939 96  3.9672 0.02714856237458
48    3 - 4   -13.84 1.6939 96 -8.1705 0.00000000079632
51    3 - 7    -7.96 1.6939 96 -4.6992 0.00211890975287
54   3 - 10    -6.84 1.6939 96 -4.0380 0.02164538214065
55   3 - 11    -8.52 1.6939 96 -5.0298 0.00058886856553
56   3 - 12    -7.68 1.6939 96 -4.5339 0.00391182008602
57   3 - 13    -7.12 1.6939 96 -4.2033 0.01252639251896
61   3 - 17    -7.60 1.6939 96 -4.4867 0.00464383583599
62   3 - 18    -6.56 1.6939 96 -3.8727 0.03643635567587
64   3 - 20    -8.16 1.6939 96 -4.8173 0.00135190254111
66   3 - 22    -7.88 1.6939 96 -4.6520 0.00252953732975
68   3 - 24    -6.52 1.6939 96 -3.8491 0.03916121121781
69   3 - 25    -7.68 1.6939 96 -4.5339 0.00391182008602
70    4 - 5    10.96 1.6939 96  6.4703 0.00000118997242
 [ reached 'max' / getOption("max.print") -- omitted 13 rows ]
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