Chapter 4 Contrasts
Analysis of Factor Level Means

- When $F$-test is significant; there exist differences among the means. Now what?
- Want to determine which means are different and identify treatments statistically of the same effect
Visual Assessment

- Can often get an idea by looking at plots
  - Side-by-side Box Plots
  - Scatter plots

These plots do not give any information about the precision of the estimates. Need to consider standard errors.
Confidence Interval for Treatment Means

Recall the models for a CRD experiment:

\[ y_{ij} = \mu_i + \varepsilon_{ij} \]  
\[ = \mu + \alpha_i + \varepsilon_{ij} \]

(Means Model)  
(Effects Model)

where \( \varepsilon_{ij} \)'s are i.i.d. \( N(0, \sigma^2) \)

In the means model, \( \mu_i \) is estimated by the sample mean of treatment group \( i \)

\[ \hat{\mu}_i = \bar{y}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} = \frac{1}{n_i} (y_{i1} + \ldots + y_{in_i}). \]

\[ \mathbb{E}(\bar{y}_{i\cdot}) = \mathbb{E} \left[ \frac{1}{n_i} (y_{i1} + \ldots + y_{in_i}) \right] = \frac{1}{n_i} \left[ \mathbb{E}(y_{i1}) + \ldots + \mathbb{E}(y_{in_i}) \right] \]

\[ = \frac{1}{n_i} (\mu_i + \ldots + \mu_i) = \mu_i. \]

So \( \bar{y}_{i\cdot} \) is an unbiased estimate of \( \mu_i \).
Confidence Interval for Treatment Means

\[
\mathrm{Var}(\bar{y}_{i\cdot}) = \mathrm{Var}\left(\frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}\right) = \frac{1}{n_i^2} \mathrm{Var}\left(\sum_{j=1}^{n_i} y_{ij}\right)
\]

\[
= \frac{1}{n_i^2} \sum_{j=1}^{n_i} \mathrm{Var}(y_{ij}) \quad \text{(by the indep. of } y_{i1}, \ldots, y_{in_i})
\]

\[
= \frac{1}{n_i^2} \sum_{j=1}^{n_i} \sigma^2 = \frac{n_i \sigma^2}{n_i^2} = \frac{\sigma^2}{n_i}
\]

So the standard deviation of \( \bar{y}_{i\cdot} \) is \( \mathrm{SD}(\bar{y}_{i\cdot}) = \sqrt{\mathrm{Var}(\bar{y}_{i\cdot})} = \frac{\sigma}{\sqrt{n_i}} \n\)

The unknown \( \sigma \) is estimated by \( \sqrt{\text{MSE}} \).
Recall the standard error is an estimate of SD, in which the unknown \( \sigma \) is replaced by some estimate \( \hat{\sigma} \). So

\[
\mathrm{SE}(\bar{y}_{i\cdot}) = \mathrm{SD}(\bar{y}_{i\cdot}) = \frac{\sqrt{\text{MSE}}}{\sqrt{n_i}}.
\]
Confidence Interval for Treatment Means

Most of the confidence intervals have the general form:

\[
\text{Confidence Interval} = \text{estimate} \pm (\text{multiplier} \times \text{standard error})
\]

The 100(1 − \(\alpha\))% Confidence Interval (C.I.) for \(\mu_i\) is

\[
\bar{y}_i \pm t_{\alpha/2, N-g} \frac{\sqrt{MSE}}{\sqrt{n_i}}
\]

For resin glue failure time example, we have \(N = 37\), \(g = 5\). For \(\alpha = 0.05\), \(1 - \alpha/2 = 0.975\), we know \(t_{37-5,0.975} \approx 2.037\).

\[> \text{qt}(1-0.05/2, \text{df} = 37-5)\]

\[\text{[1]} \ 2.036933\]

We have also found \(MSE = 0.294\) and hence

<table>
<thead>
<tr>
<th>Temp</th>
<th>(n_i)</th>
<th>(\bar{y}_i)</th>
<th>95% C.I. for (\mu_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>175</td>
<td>8</td>
<td>1.933</td>
<td>1.933 ± 2.037 × (\sqrt{0.294/8}) = 1.933 ± 0.390</td>
</tr>
<tr>
<td>194</td>
<td>8</td>
<td>1.629</td>
<td>1.629 ± 2.037 × (\sqrt{0.294/8}) = 1.629 ± 0.390</td>
</tr>
<tr>
<td>213</td>
<td>8</td>
<td>1.378</td>
<td>1.378 ± 2.037 × (\sqrt{0.294/8}) = 1.378 ± 0.390</td>
</tr>
<tr>
<td>231</td>
<td>7</td>
<td>1.194</td>
<td>1.194 ± 2.037 × (\sqrt{0.294/7}) = 1.194 ± 0.417</td>
</tr>
<tr>
<td>250</td>
<td>6</td>
<td>1.057</td>
<td>1.057 ± 2.037 × (\sqrt{0.294/6}) = 1.057 ± 0.451</td>
</tr>
</tbody>
</table>

Chapter 4 - 6
Consider the general pairwise comparison (difference between two means): $\mu_i - \mu_k$

- the estimator is $\bar{y}_{i\bullet} - \bar{y}_{k\bullet}$
- Since $\bar{y}_{i\bullet}$ and $\bar{y}_{k\bullet}$ are independent,

$$\text{Var}(\bar{y}_{i\bullet} - \bar{y}_{k\bullet}) = \text{Var}(\bar{y}_{i\bullet}) + \text{Var}(\bar{y}_{k\bullet}) = \frac{\sigma^2}{n_i} + \frac{\sigma^2}{n_k}$$

- $\text{SD}(\bar{y}_{i\bullet} - \bar{y}_{k\bullet}) = \sqrt{\text{Var}(\hat{\mu}_i - \hat{\mu}_k)} = \sqrt{\sigma^2 \left( \frac{1}{n_i} + \frac{1}{n_k} \right)}$,

- $\text{SE}(\bar{y}_{i\bullet} - \bar{y}_{k\bullet}) = \hat{\text{SD}}(\bar{y}_{i\bullet} - \bar{y}_{k\bullet}) = \sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_k} \right)}$. 

Chapter 4 - 7
Confidence Intervals for Differences

The 100(1 − α)% confidence interval (C.I.) for $\mu_i − \mu_k$ is

$$
\bar{y}_i \mp \bar{y}_k \pm t_{\alpha/2, N−g} \sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_k} \right)}
$$

Note this is NOT equivalent to a two-sample comparison between treatment $i$ and treatment $k$, which is

$$
\bar{y}_i \mp \bar{y}_k \pm t_{\alpha/2, n_i+n_k−2} \sqrt{s_p^2 \left( \frac{1}{n_i} + \frac{1}{n_k} \right)}
$$

- MSE = \( \sum_{i=1}^{g} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i \cdot)^2 \) \( \frac{1}{N−g} \) is a more accurate estimator of $\sigma^2$ than the pooled sample variance

$$
s_p^2 = \frac{\sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i \cdot)^2 + \sum_{j=1}^{n_k} (y_{kj} - \bar{y}_k \cdot)^2}{n_i + n_k − 2}
$$

- The multiplier for the two-sample C.I. is larger

$$
t_{\alpha/2, n_i+n_k−2} > t_{\alpha/2, N−g}
$$
Hypothesis Testing for Difference

For the hypotheses

\[
\begin{cases}
H_0 : \mu_i - \mu_k = 0 \\
H_a : \mu_i - \mu_k \neq 0
\end{cases}
\]

the test statistic is

\[
t_0 = \frac{\bar{y}_i - \bar{y}_k}{\text{SE}(\bar{y}_i - \bar{y}_k)} = \frac{\bar{y}_i - \bar{y}_k}{\sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_k} \right)}} \sim t_{N-g}
\]

which is of \( t \)-distribution with d.f. \( = N - g \). The \( P \)-value is

\[
P(\mid t_{N-g} \mid > \mid t_0 \mid) = 2P(t_{N-g} > \mid t_0 \mid).
\]

For one-sided alternatives, the \( P \)-value is

\[
P(t_{N-g} > t_0), \quad \text{if } H_a : \mu_i - \mu_k > 0,
\]

\[
P(t_{N-g} < t_0), \quad \text{if } H_a : \mu_i - \mu_k < 0.
\]
Example — Beet Lice

- Goal: efficacy of 4 chemical treatments for beet lice
- Experimental units: 100 beet plants in individual pots in total, 25 plants per treatment, randomly assigned
- Response: # of lice on each plant at the end of the 2nd week
- The plots are spatially separated (in this random order)

```r
> beet = read.table("beetlice.txt", header=TRUE)
> attach(beet)
> plot(ttt,licecount,xlab="Treatment",ylab="Number of Lice")
```

![Box plot of lice counts by treatment]

Chapter 4 - 10
Example — Beet Lice

The group means of the 4 treatments are

\[
> \text{tapply(licecount, ttt, mean)} \\
\begin{array}{cccc}
A & B & C & D \\
12.00 & 14.96 & 18.36 & 24.00 \\
\end{array}
\]

From the ANOVA table below, we get MSE = 47.8.

\[
> \text{aov1 = aov(licecount ~ ttt, data = beet)} \\
> \text{summary(aov1)} \\
\begin{array}{cccccc}
Df & Sum Sq & Mean Sq & F value & Pr(>F) \\
ttt & 3 & 1989 & 663.1 & 13.86 & 1.39e-07 *** \\
Residuals & 96 & 4593 & 47.8 & \\
\end{array}
\]

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

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Example — Beet Lice

<table>
<thead>
<tr>
<th>Chemical</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>( \text{MSE} = 47.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{y}_i )</td>
<td>12.00</td>
<td>14.96</td>
<td>18.36</td>
<td>24.00</td>
<td></td>
</tr>
</tbody>
</table>

Say we want to compare chemical A and B, \( \mu_B - \mu_A \). The estimate is \( \bar{y}_B - \bar{y}_A = 14.96 - 12.00 = 2.96 \) with SE

\[
\text{SE}(\bar{y}_B - \bar{y}_A) = \sqrt{\text{MSE}} \sqrt{\frac{1}{n_A} + \frac{1}{n_B}} = \sqrt{47.8} \sqrt{\frac{1}{25} + \frac{1}{25}} = 1.956
\]

A 95% confidence interval for \( \mu_B - \mu_A \) is

\[
\bar{y}_B - \bar{y}_A \pm t_{96,0.025} \times \text{SE}(\bar{y}_B - \bar{y}_A)
\]

\[
= 2.96 \pm 1.984984 \times 1.956 = ( -0.923, 6.843 )
\]

in which \( t_{96,0.025} \) is found using the R command

\[
> \text{qt}(0.975, \text{df}=96) \\
[1] 1.984984
\]

The two chemicals are not significantly different at 5% level because 0 is in the 95% confidence interval.
Example — Beet Lice

If one wants to test whether chemical A and chemical B have the same effect

\[
\begin{align*}
H_0 & : \mu_B - \mu_A = 0 \\
H_a & : \mu_B - \mu_A \neq 0
\end{align*}
\]

the test statistic is

\[
t_0 = \frac{\bar{y}_B - \bar{y}_A}{SE(\bar{y}_B - \bar{y}_A)} = \frac{2.96}{1.956} = 1.513
\]

with \(100 - 4 = 96\) degrees of freedom. The \(P\)-value is

\[
> 2*pt(-1.513, \text{ df = 96})
\]

[1] 0.1335649

As the \(P\)-value > 0.05, we again confirm that the two chemicals are similar in efficacy.
Example — Beet Lice

```
> lmA = lm(licecount ~ as.factor(ttt), data=beet)
> summary(lmA)
...(some output omitted)

Coefficients:

|                     | Estimate | Std. Error | t value | Pr(>|t|) |
|---------------------|----------|------------|---------|----------|
| (Intercept)         | 12.000   | 1.383      | 8.675   | 1.04e-13 *** |
| as.factor(ttt)B     | 2.960    | 1.956      | 1.513   | 0.13356  |
| as.factor(ttt)C     | 6.360    | 1.956      | 3.251   | 0.00159 ** |
| as.factor(ttt)D     | 12.000   | 1.956      | 6.134   | 1.91e-08 *** |

---

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.917 on 96 degrees of freedom
Multiple R-squared: 0.3022,  Adjusted R-squared: 0.2804
F-statistic: 13.86 on 3 and 96 DF,  p-value: 1.394e-07
```
Example — Beet Lice

```r
> beet$ttt = relevel(as.factor(beet$ttt), ref = 2)
> lmB = lm(licecount ~ as.factor(ttt), data=beet)
> summary(lmB)
...(some output omitted)
```

Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| (Intercept) | 14.960 | 1.383 | 10.814 | < 2e-16 *** |
| as.factor(ttt)A | -2.960 | 1.956 | -1.513 | 0.1336 |
| as.factor(ttt)C | 3.400 | 1.956 | 1.738 | 0.0854 . |
| as.factor(ttt)D | 9.040 | 1.956 | 4.621 | 1.19e-05 *** |

---

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.917 on 96 degrees of freedom
Multiple R-squared: 0.3022, Adjusted R-squared: 0.2804
F-statistic: 13.86 on 3 and 96 DF, p-value: 1.394e-07

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Example — Beet Lice

```r
> beet$ttt = relevel(as.factor(beet$ttt), ref = 3)
> lmC = lm(licecount ~ as.factor(ttt), data=beet)
> summary(lmC)
... (some output omitted)
```

Coefficients:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| (Intercept) | 18.360 | 1.383 | 13.272 | < 2e-16 *** |
| as.factor(ttt)B | -3.400 | 1.956 | -1.738 | 0.08543 . |
| as.factor(ttt)A | -6.360 | 1.956 | -3.251 | 0.00159 ** |
| as.factor(ttt)D | 5.640 | 1.956 | 2.883 | 0.00486 ** |

---

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.917 on 96 degrees of freedom
Multiple R-squared: 0.3022, Adjusted R-squared: 0.2804
F-statistic: 13.86 on 3 and 96 DF, p-value: 1.394e-07
Pairwise Comparison Summary

<table>
<thead>
<tr>
<th>Chemical Comparison</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_B - \mu_A$</td>
<td>2.960</td>
<td>1.956</td>
<td>1.513</td>
<td>0.13356</td>
</tr>
<tr>
<td>$\mu_C - \mu_A$</td>
<td>6.360</td>
<td>1.956</td>
<td>3.251</td>
<td>0.00159 **</td>
</tr>
<tr>
<td>$\mu_D - \mu_A$</td>
<td>12.000</td>
<td>1.956</td>
<td>6.134</td>
<td>1.91 × 10^{-8} ***</td>
</tr>
<tr>
<td>$\mu_C - \mu_B$</td>
<td>3.400</td>
<td>1.956</td>
<td>1.738</td>
<td>0.0854  .</td>
</tr>
<tr>
<td>$\mu_D - \mu_B$</td>
<td>9.040</td>
<td>1.956</td>
<td>4.621</td>
<td>1.19 × 10^{-5} ***</td>
</tr>
<tr>
<td>$\mu_D - \mu_C$</td>
<td>5.640</td>
<td>1.956</td>
<td>2.883</td>
<td>0.00486 **</td>
</tr>
</tbody>
</table>

Display Results — Underline Diagram (p.88, Section 5.4.1)
1. Write out treatment labels horizontally in increasing order sorted by treatment means
2. Draw a line segment under a group of treatments if no pair of treatments in that group is significantly different.

A B C D
4.1-4.2 Contrasts

Contrasts is a more general form of treatment comparisons.

A **contrast** is a linear combination of group means $\mu_i$’s

$$C = \sum_{i=1}^{g} \omega_i \mu_i$$ such that $\sum_{i=1}^{g} \omega_i = 0$

- **Every pairwise comparison is a contrast!** ($C = \mu_i - \mu_k$)
  $\omega_i = 1$, $\omega_k = -1$, all other $\omega_i$’s are 0, and
  $\sum_{i=1}^{g} \omega_i = 1 + (-1) + 0 + \cdots + 0 = 0$

- **A single treatment mean $C = \mu_i$ is NOT a contrast**

- **Is $C = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5}{3}$ a contrast?** Yes.
  $\omega_1 = \omega_2 = \frac{1}{2}$, $\omega_3 = \omega_4 = \omega_5 = -\frac{1}{3}$, which adds up to 0.

- **Is $C = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4}{2} + \mu_5$ a contrast?** No.
  $\omega_1 = \omega_2 = \frac{1}{2}$, $\omega_3 = \omega_4 = -\frac{1}{2}$, $\omega_5 = 1$, which add up to 1, not 0.
Recall

the means model: \( y_{ij} = \mu_i + \varepsilon_{ij} \)

and the effects model: \( y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \).

In the effects model, the “treatment effect” \( \alpha_i = \mu_i - \mu \) is arbitrary, because it depends on how the grand mean \( \mu \) was defined, e.g.,

\[
\mu = \mu_1, \quad \text{or} \quad \mu = \frac{1}{N} \sum_{i=1}^{g} n_i \mu_i, \quad \text{or} \quad \mu = \frac{1}{g} \sum_{i=1}^{g} \mu_i.
\]

If possible, avoid talking about the size of a treatment effect without clearly stating the definition of \( \mu \) and \( \alpha_i \)'s.

Contrasts make comparisons and do not have the problem of effects depending on an arbitrary definition of \( \mu \).

\[
C = \sum_{i=1}^{g} \omega_i \mu_i = \sum_{i=1}^{g} \omega_i (\mu + \alpha_i) = \mu \sum_{i=1}^{g} \omega_i + \sum_{i=1}^{g} \omega_i \alpha_i = \sum_{i=1}^{g} \omega_i \alpha_i = 0
\]
Estimator and Confidence Interval for a Contrast

A natural estimator for a contrast \( C = \sum_{i=1}^{g} \omega_i \mu_i \) is

\[
\hat{C} = \sum_{i=1}^{g} \omega_i \bar{y}_i.
\]

By the independence of \( \bar{y}_i, i = 1, \ldots, g \), we know

\[
\text{Var}\left( \sum_{i=1}^{g} \omega_i \bar{y}_i \right) = \sum_{i=1}^{g} \text{Var}(\omega_i \bar{y}_i) = \sum_{i=1}^{g} \omega_i^2 \text{Var}(\bar{y}_i) = \sum_{i=1}^{g} \omega_i^2 \frac{\sigma^2}{n_i}.
\]

The estimator \( \hat{C} \) has standard deviation and standard error

\[
\text{SD}(\hat{C}) = \sqrt{\sigma^2 \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}, \quad \text{SE}(\hat{C}) = \sqrt{\text{MSE} \times \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}.
\]

A \((1 - \alpha)100\%\) confidence interval for the contrast \( C \) is

\[
\hat{C} \pm t_{\alpha/2,N-g} \times \text{SE}(\hat{C})
\]
Hypothesis Testing for a Contrast

For the hypotheses

\[
\begin{align*}
H_0 & : \quad C = 0 \\
H_a & : \quad C \neq 0
\end{align*}
\]

the test statistic is

\[
t_0 = \frac{\hat{C}}{SE(\hat{C})} = \frac{\sum_{i=1}^{g} \omega_i \bar{y}_i \cdot}{\sqrt{\text{MSE} \times \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}} \sim t_{N-g}
\]

which is of \( t \)–distribution with d.f. = \( N - g \). For a two-sided test, the \( P \)-value is

\[2P(t_{N-g} > |t_0|)\]

For one-sided alternative, the one-sided \( P \)-value is

\[
\begin{align*}
P(t_{N-g} > t_0), & \quad \text{if } H_a : C > 0, \\
P(t_{N-g} < t_0), & \quad \text{if } H_a : C < 0.
\end{align*}
\]
Example — Beet Lice

<table>
<thead>
<tr>
<th>Chemical</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{y}_i$</td>
<td>12.00</td>
<td>14.96</td>
<td>18.36</td>
<td>24.00</td>
</tr>
</tbody>
</table>

We want a comparison between treatment A, B, C (all liquid) and treatment D (powder). We thus consider the contrast

$$C = \frac{\mu_1 + \mu_2 + \mu_3}{3} - \mu_4$$

in which $(\omega_1, \omega_2, \omega_3, \omega_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. The contrast is estimated by

$$\hat{C} = \frac{\bar{y}_A + \bar{y}_B + \bar{y}_C}{3} - \bar{y}_D = \frac{12.00 + 14.96 + 18.36}{3} - 24.00 = -8.893$$

with standard error

$$SE(\hat{C}) = \sqrt{\frac{MSE}{\sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}} = \sqrt{47.8 \left( \frac{(1/3)^2}{25} + \frac{(1/3)^2}{25} + \frac{(1/3)^2}{25} + \frac{(-1)^2}{25} \right)} = \sqrt{47.8 \times \frac{4}{75}} = 1.597.$$
Example — Beet Lice

The 95% confidence interval for $C$ is

$$\hat{C} \pm t_{0.025,96} \times \text{SE}(\hat{C})$$

$$= -8.893 \pm 1.984984 \times 1.597 = (-12.063, -5.723)$$

in which $t_{0.025,96}$ is found in R via the command

```r
> qt(0.975, df=96)
[1] 1.984984
```

The $t$-statistic is

$$t_0 = \frac{\hat{C}}{\text{SE}(\hat{C})} = \frac{-8.893}{1.597} = -5.568,$$

with $100 - 4 = 96$ degrees of freedom. The two-sided $p$-value is

```r
> 2*pt(-5.568, df=96)
[1] 2.339348e-07
```

Because 0 is not in the confidence interval, or because of the small $P$-value, we can conclude that the 3 liquid chemicals are less effective than chemical D.
Two contrasts $C = \sum_{i=1}^{g} \omega_i \mu_i$ and $D = \sum_{i=1}^{g} \omega_i^* \mu_i$ are said to be orthogonal if

$$\sum_{i=1}^{g} \frac{\omega_i \omega_i^*}{n_i} = 0.$$ 

Note if the design is balanced $n_1 = n_2 = \cdots = n_g$, the condition is equivalent to $\sum_{i=1}^{g} \omega_i \omega_i^* = 0$.

When two contrasts are orthogonal, their estimators are independent.