TOPIC. Moment generating functions. The n^{th} ***moment*** of a random variable X is $E(X^n)$ (if this quantity exists); the ***moment generating function (MGF)*** of X is the function defined by

$$M(t) := E(e^{tX}) \tag{1}$$

for $t \in \mathbb{R}$; the expectation in (1) exists (since e^{tX} is a nonnegative) but may be $+\infty$. This section shows how the MGF "generates" the moments, and how the moments can (in certain cases) be used to recover the MGF.

Note that if $Y = a + bX$ for numbers a and b, then

$$M_Y(t) := E(e^{tY}) = E(e^{at}e^{btX}) = e^{at}E(e^{btX}) = e^{at}M_X(bt) \tag{2}$$

for all $t \in \mathbb{R}$. Consequently, given a location/scale family of distributions, it suffices to compute the MGF for one (simple) member in the family; the MGFs of the other members follow from (2).

Example 1. The mgfs in this example are illustrated in Figure 1.

(a) Suppose $X \sim N(0,1)$. Then

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\ &= e^{t^2/2}\int_{-\infty}^{\infty}\frac{e^{-t^2/2+tx-x^2/2}}{\sqrt{2\pi}}\,dx = e^{t^2/2}\int_{-\infty}^{\infty}\frac{e^{-(x-t)^2/2}}{\sqrt{2\pi}}\,dx = e^{t^2/2}. \end{aligned} \tag{3}$$

In particular, $\{\, t \in \mathbb{R} : M(t) < \infty \,\} = \mathbb{R}$.

(b) Suppose $X \sim \text{Gamma}(r,1)$. Then

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_0^{\infty} e^{tx}\frac{x^{r-1}e^{-x}}{\Gamma(r)}\,dx \\ &= \frac{1}{(1-t)^r}\int_0^{\infty}\frac{(1-t)^r x^{r-1}e^{-(1-t)x}}{\Gamma(r)}\,dx = \frac{1}{(1-t)^r} \end{aligned} \tag{4}$$

for $t < 1$; however $M(t) = \infty$ for $t \geq 1$. In particular, $\{\, t \in \mathbb{R} : M(t) < \infty \,\} = (-\infty, 1)$.

(c) Suppose $X \sim UF(2,2)$, with density $f(x) = 1/(1+x)^2$ for $x \geq 0$ and 0 otherwise. Then

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_0^{\infty} e^{tx}\frac{1}{(1+x)^2}\,dx \\ &= e^{-t}\int_1^{\infty}\frac{e^{ty}}{y^2}\,dy = \begin{cases} e^{-t}E_2(-t), & \text{if } t < 0, \\ 1, & \text{if } t = 0, \\ \infty, & \text{if } t > 0; \end{cases} \end{aligned} \tag{5}$$

here $E_n(u) := \int_1^{\infty} e^{-uy}/y^n\,dx$ is the value of the so-called ***exponential integral*** of order n at u. In particular, $\{\, t \in \mathbb{R} : M(t) < \infty \,\} = (-\infty, 0]$. (In MAPLE, $E_n(u)$ is `Ei(n,u)`.)

(d) Suppose $X \sim \text{Cauchy}(0,1)$. Then

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\pi(1+x^2)}\,dx = \begin{cases} 1, & \text{if } t = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{6}$$

Here $\{\, t \in \mathbb{R} : M(t) < \infty \,\} = \{0\}$. •

These examples motivate the following result.

Theorem 1. *Let M be the MGF of a random variable X. Then*

$$B := \{t \in \mathbb{R} : M(t) < \infty\} \tag{7}$$

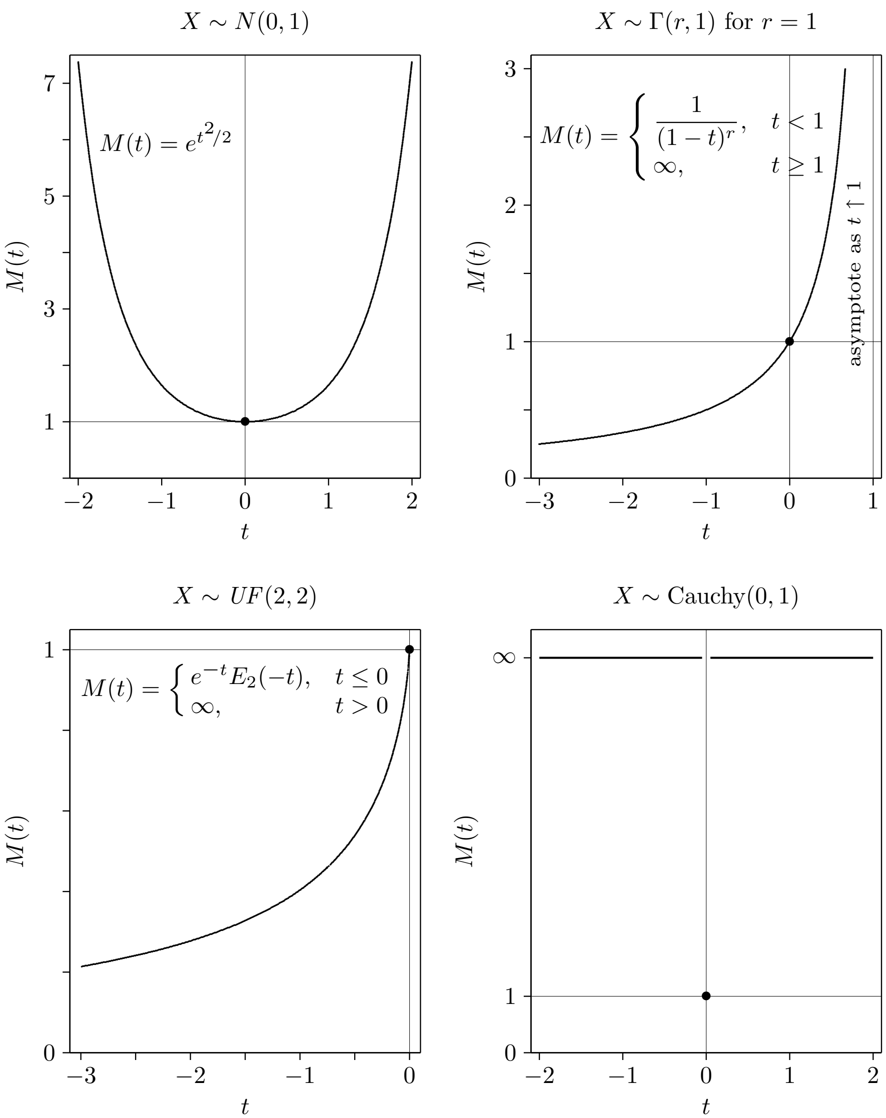
is a (perhaps degenerate) interval containing 0. *Moreover M is convex on B.*

Proof The exponential function $x \rightsquigarrow e^x$ is convex on $\mathbb{R}$. Hence for t_1 and t_2 in $\mathbb{R}$ and $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} & e^{(\alpha t_1 + (1-\alpha)t_2)X(\omega)} \leq \alpha e^{t_1 X(\omega)} + (1-\alpha)e^{t_2 X(\omega)} \text{ for all } \omega \\ & \quad \Longrightarrow E\big(e^{(\alpha t_1 + (1-\alpha)t_2)X}\big) \leq E\big(\alpha e^{t_1 X} + (1-\alpha)e^{t_2 X}\big) \\ & \quad \Longrightarrow M\big(\alpha t_1 + (1-\alpha)t_2\big) \leq \alpha M(t_1) + (1-\alpha)M(t_2). \end{aligned} \tag{8}$$

Consequently if $M(t_1) < \infty$ and $M(t_2) < \infty$, then $M(t) < \infty$ for all $t \in [t_1, t_2]$; it follows that B is an interval. B contains 0 because $M(0) = E(e^{0X}) = E(e^0) = 1$. (8) says that M is convex on B. ∎

Figure 1: Graphs of $M(t) = E(e^{tX})$ for the random variables X in Example 1.



The derivatives of the MGF. Let X be a random variable with MGF M. Let $B = \{t : M(t) < \infty\}$ be the interval of finiteness of M, and let B° be the interior of B. With luck, we should have

$$\frac{d}{dt}M(t) = \frac{d}{dt}E(e^{tX}) \stackrel{?}{=} E\Big(\frac{d}{dt}e^{tX}\Big) = E\big(Xe^{tX}\big)$$

for $t \in B^\circ$. The following theorem asserts that this is correct.

Theorem 2. *Let X, M, B, and B° be as above, and suppose that B° is nonempty. Then for each $t \in B^\circ$, M is differentiable at t, the random variable Xe^{tX} is integrable, and*

$$M'(t) = E\big(Xe^{tX}\big). \tag{9}$$

Proof Let $t_0 \in B^\circ$. Let $\epsilon > 0$ be such that the points $t_0 - 2\epsilon$ and $t_0 + 2\epsilon$ are both in B. Let $t_1, t_2, \ldots$ be points in B° such that $t_n \to t_0$ as $n \to \infty$ and $0 < |t_n - t_0| \leq \epsilon$ for all n. For each n

$$\frac{M(t_n) - M(t_0)}{t_n - t_0} = \frac{E(e^{t_n X}) - E(e^{t_0 X})}{t_n - t_0} = E(Y_n)$$

where

$$Y_n = \frac{e^{t_n X} - e^{t_0 X}}{t_n - t_0}.$$

As $n \to \infty$, we have

$$Y_n \to \frac{d}{dt}e^{tX}\Big|_{t=t_0} = Xe^{t_0 X} := Y.$$

According to the DCT, provided there exists

$$\text{an integrable rv } D \text{ such that } |Y_n| \leq D \text{ for all } n, \tag{10}$$

the Y_n's and Y will be integrable and we will have

$$E(Y_n) \to E(Y).$$

This is exactly the conclusion we want: it implies that $Xe^{t_0 X}$ is integrable and that $M'(t_0) = E(Xe^{t_0 X})$.

(10): D is integrable and $|Y_n| \le D$ for all n. $\qquad t_0 \pm 2\epsilon \in B$

We need to produce a D satisfying (10). By the mean value theorem,

$$e^{t_n X} - e^{t_0 X} = (t_n - t_0)\, X e^{t^* X}$$

for some point t^* between t_0 and t_n; t^* depends on n and the implicit sample point ω, but we don't indicate that in the notation. Hence

$$|Y_n| = \left|\frac{e^{t_n X} - e^{t_0 X}}{t_n - t_0}\right| = |X| e^{t^* X} \le |X|\big(e^{(t_0-\epsilon)X} + e^{(t_0+\epsilon)X}\big).$$

Now

$$\begin{aligned}|X| &= \frac{1}{\epsilon}\epsilon|X| \le \frac{1}{\epsilon}\Big(1 + \epsilon|X| + \frac{\epsilon^2|X|^2}{2!} + \frac{\epsilon^3|X|^3}{3!} + \cdots\Big)\\ &= \frac{1}{\epsilon}e^{\epsilon|X|} \le \frac{1}{\epsilon}\big(e^{-\epsilon X} + e^{\epsilon X}\big).\end{aligned}$$

Thus

$$\begin{aligned}|Y_n| &\le \frac{1}{\epsilon}\big(e^{-\epsilon X} + e^{\epsilon X}\big)\,\big(e^{(t_0-\epsilon)X} + e^{(t_0+\epsilon)X}\big)\\ &\le D := \frac{1}{\epsilon}\big(e^{(t_0-2\epsilon)X} + 2e^{t_0 X} + e^{(t_0+2\epsilon)X}\big).\end{aligned}$$

We have

$$E(D) = \frac{1}{\epsilon}\big(M(t_0-2\epsilon) + 2M(t_0) + M(t_0+2\epsilon)\big) < \infty$$

by the choice of ϵ. Hence D is the integrable dominator we need. ∎

Iterating this argument (do it!) gives

Theorem 3. *Let X, M, B, B° be as in Theorem 2, and suppose that B° is nonempty. Then M is infinitely differentiable on B°. For each $t \in B^\circ$ and each $k \in \mathbb{N}$, $X^k e^{tX}$ is integrable and*

$$M^{(k)}(t) = E\big(X^k e^{tX}\big). \tag{11}$$

In particular if $0 \in B^\circ$, then for all $k \in \mathbb{N}$, X^k is integrable and

$$M^{(k)}(0) = E(X^k). \tag{12}$$

$0 \in B^\circ \implies M^{(k)}(0)$ exists and equals $E(X^k)$, for all $k \in \mathbb{N}$.

Example 2. Suppose $X \sim \text{Gamma}(r,1)$. By (4)

$$M(t) = \frac{1}{(1-t)^r}$$

for $t < 1$. Consequently $0 \in B^\circ = (-\infty, 1)$ and

$$\begin{aligned}E(X^k) = M^{(k)}(0) &= \frac{d^{k-1}}{dt^{k-1}}\Big(\frac{r}{(1-t)^{r+1}}\Big)\Big|_{t=0}\\ &= \frac{r(r+1)(r+2)\cdots(r+k-1)}{(1-t)^{r+k}}\Big|_{t=0}\\ &= r(r+1)(r+2)\cdots(r+k-1) = \frac{\Gamma(r+k)}{\Gamma(r)}.\end{aligned} \tag{13}$$

•

A related but different argument (see Exercise 5) gives this companion to Theorem 3:

Theorem 4. *Let X, M, B, and B° be as in Theorem 3. Suppose that $0 \notin B^\circ$, but that there exists an $\epsilon > 0$ such that $[0, \epsilon) \subset B$. Then for all $k \in \mathbb{N}$, X^k is quasi-integrable and*

$$E(X^k) = M_+^{(k)}(0) := \lim_{t \downarrow 0} \frac{M^{(k-1)}(t) - M_+^{(k-1)}(0)}{t}. \tag{14}$$

$M_+^{(k)}(0)$ is called the k^{th} right-hand derivative of M at 0; by definition $M_+^{(0)}(0) = M(0) = 1$. A similar result holds for the left-hand derivatives $M_-^{(k)}(0)$ of M at 0 when $(-\epsilon, 0] \subset B$.

Example 3. Suppose $X \sim UF(2,2)$ as in Example 1 (c). Since $B = (-\infty, 0]$, we have

$$M_-^{(k)}(0) = E(X^k) = \int_0^\infty \frac{x^k}{(1+x)^2}\,dx = \infty.$$

for all $k \in \mathbb{N}$. •

The series expansion of the MGF. Consider the following heuristic

$$M(t) = E(e^{tX}) = E\Big(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\Big) \overset{?}{=} \sum_{k=0}^{\infty} E(X^k)\frac{t^k}{k!}. \qquad (15)$$

This couldn't be true in general, because the power series on the RHS converges to a finite value in an interval which is symmetric about 0, whereas the domain of finiteness of M can be an asymmetric interval about 0.

Theorem 5. *Let X be a random variable with MGF M. Then the following two statements are equivalent:*

- S1 *X^k is integrable for all $k \in \mathbb{N}$ and the series $\sum_{k=0}^{\infty} t^k E(X^k)/k!$ has radius of convergence $R > 0$.*
- S2 $R^* := \sup\{\, t > 0 : M(t) < \infty$ and $M(-t) < \infty \,\} > 0$.

If S1 *and* S2 *hold, then*

- S3 *$R = R^*$, and*
- S4 *$M(t) = \sum_{k=0}^{\infty} t^k E(X^k)/k!$ for all t such that $M(t) < \infty$ and $M(-t) < \infty$, and in particular for all t such $|t| < R^* = R$.*

Example 4. (a) Suppose $X \sim \text{Gamma}(r, 1)$. Then $B = (-\infty, 1) \Longrightarrow R^* = 1 \Longrightarrow R = 1 \Longrightarrow$ (15) holds for $|t| < 1$. Note that the LHS of (15) is defined and finite for $t < -1$, even though the series on the RHS does not converge for such t's.

(b) Suppose $X \sim N(0, 1)$. Then $B = (-\infty, \infty)$, so $R = R^* = \infty$. Consequently X^k is integrable for all $k \in \mathbb{N}$ and

$$\sum_{n=0}^{\infty} E(X^n)\frac{t^n}{n!} = M(t) = e^{t^2/2}$$
$$= \sum_{k=0}^{\infty} \frac{(t^2/2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,2^k}\frac{t^{2k}}{(2k)!}.$$

S1: X^k is integrable for all $k \in \mathbb{N}$ and the series $\sum_{k=0}^{\infty} t^k E(X^k)/k!$ has radius of convergence $R > 0$.

S2: $R^* := \sup\{\, t > 0 : M(t) < \infty$ and $M(-t) < \infty \,\} > 0$.

S3: $R = R^*$. S4: $M(t) = \sum_{k=0}^{\infty} E(X^k)t^k/k!$ for $|t| < R = R^*$.

By the uniqueness of the coefficients of a power series, $E(X^n) = 0$ for n odd, whereas

$$E(X^{2k}) = \frac{(2k)!}{k!\,2^k} = \frac{1 \cdot 2 \cdot 3 \cdots (2k)}{2 \cdot 4 \cdot 6 \cdots (2k)} = 1 \cdot 3 \cdot 5 \cdots (2k-1) \qquad (16)$$

is the product of the odd positive integers less than $2k$. In particular,

$$E(X^2) = 1, \qquad E(X^6) = 1 \cdot 3 \cdot 5 = 15,$$
$$E(X^4) = 1 \cdot 3 = 3, \quad E(X^8) = 1 \cdot 3 \cdot 5 \cdot 7 = 105\,.$$

Remember these formulas!

(c) Suppose S2 holds. Then $M(t) = \sum_{k=0}^{\infty} E(X^k)t^k/k!$ for all t with $|t| < R = R^*$. It follows from the theory of power series that M is infinitely differentiable in the interval $(-R, R)$ and that $M^{(k)}(0) = E(X^k)$ for all $k \in \mathbb{N}$. This gives another proof of (12). •

Proof of Theorem 5. • *Step 1: If $t \neq 0$ and $M(\pm t) < \infty$, then X^k is integrable for all k and $M(t) = \sum_{k=0}^{\infty} E(X^k)t^k/k!$.* Indeed, we have

$$E\Big(\sum_{k=0}^{\infty}\Big|\frac{t^k X^k}{k!}\Big|\Big) = E(e^{|tX|}) \le E(e^{tX} + e^{-tX})$$
$$= M(t) + M(-t) < \infty.$$

In particular, $E\big(|t^k||X^k|/k!\big) < \infty \Longrightarrow E(|X^k|) < \infty \Longrightarrow E(X^k)$ exists and is finite, for all $k \in \mathbb{N}$. Moreover by Fubini's Theorem

$$M(t) = E\Big(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\Big) = \sum_{k=0}^{\infty} \frac{t^k E(X^k)}{k!}\,.$$

S1: X^k is integrable for all $k \in \mathbb{N}$ and the series $\sum_{k=0}^{\infty} t^k E(X^k)/k!$ has radius of convergence $R > 0$.

S2: $R^* := \sup\{\, t > 0 : M(t) < \infty \text{ and } M(-t) < \infty \,\} > 0$.

S3: $R = R^*$. S4: $M(t) = \sum_{k=0}^{\infty} E(X^k) t^k/k!$ for $|t| < R = R^*$.

• *Step 2:* S2 $\Longrightarrow$ S1 *and* $R \geq R^*$. Suppose $0 < |t| < R^*$. Then $M(t) < \infty$ and $M(-t) < \infty$. By Step 1, X^k is integrable for all $k \in \mathbb{N}$ and $M(t) = \sum_{k=0}^{\infty} t^k E(X^k)/k!$. Since this series converges for all t with $|t| < R^*$, its radius of convergence R must be at least $R^* > 0$.

• *Step 3:* S1 $\Longrightarrow$ S2 *and* $R^* \geq R$. Suppose $0 \leq |t| < R$. Then

$$
\begin{aligned}
M(t) + M(-t) &= E(e^{tX}) + E(e^{-tX}) = E(e^{tX} + e^{-tX}) \\
&= E\Big(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-tX)^k}{k!}\Big) \\
&= E\Big(2\sum_{k=0}^{\infty} \frac{t^{2k}X^{2k}}{(2k)!}\Big) \\
&= 2\sum_{k=0}^{\infty} \frac{t^{2k}E(X^{2k})}{(2k)!} \qquad \text{(by Fubini)} \\
&\leq 2\sum_{n=0}^{\infty} \frac{|t^n E(X^n)|}{n!} \\
&< \infty,
\end{aligned}
$$

the last step holding since a power series converges absolutely in the interior of its interval of convergence. It follows that $R^* \geq R \; (> 0)$.

• *Step 4:* S1 *and/or* S2 $\Longrightarrow$ S3. This follows from Steps 2 and 3.

• *Step 5:* S1 *and/or* S2 $\Longrightarrow$ S4. This follows from Step 1. ∎

Exercise 1. Suppose $X_1, \ldots, X_n$ are independent random variables. Put $S_n = X_1 + \cdots + X_n$. Show that

$$
E(e^{tS_n}) = \prod_{i=1}^{n} E(e^{tX_i}) \tag{17}
$$

for all $t \in \mathbb{R}$. ◇

Exercise 2. Suppose $U \sim \text{Uniform}(0,1)$. Show that

$$
E(e^{tU}) = \frac{e^t - 1}{t} \tag{18}
$$

for all t (evaluate the RHS via l'Hospital's rule for $t = 0$). Use (18) and (12) (or S4) to calculate the mean and variance of U. ◇

Exercise 3. Suppose $X \sim \text{Poisson}(\lambda)$, so $P[X = k] = e^{-\lambda}\lambda^k/k!$ for $k = 0, 1, 2, \ldots$. Show that

$$
E(e^{tX}) = \exp\big(\lambda(e^t - 1)\big) \tag{19}
$$

for all $t \in \mathbb{R}$. Use (19) and (12) (or S4) to calculate the mean and variance of X. ◇

Exercise 4. Let X be a rv with density $f(x) = e^{-e^{-x}} e^{-x}$ for $x \in \mathbb{R}$. (This distribution arises in the theory of extreme values.) Show that $E(e^{tX}) = \Gamma(1 - t)$ for $t < 1$, and $= \infty$ otherwise. Express the mean and variance of X in terms of the Gamma function and its derivatives. ◇

Exercise 5. (a) Let $h: \mathbb{R} \to \mathbb{R}$ be the function which maps u to $(e^u - 1)/u$. Show that h is nondecreasing in u. (b) Use the MCT to prove (14) for the case $k = 1$. ◇

Exercise 6. As in Example 1 (c), suppose $X \sim UF(2,2)$ and put $M(t) = E(e^{tX})$. Show that

$$\frac{1 - M(-u)}{u} = \log(1/u) + \int_0^\infty \log(y)\, e^{-y}\, dy + o(1) \tag{20}$$

as $u \downarrow 0$. [Hint: Write $1 - M(-u)$ as $M(0) - M(-u)$, integrate by parts, make the change variables $y = u(1+x)$, and integrate by parts again.] ◇

Exercise 7. Suppose X has finite moments of all orders. Show that the radius R of convergence of the series in (15) is given by the formula

$$\begin{aligned} 1/R &= \limsup_n \sqrt[n]{|E(X^n)|/n!} \\ &= \limsup_n e\, \sqrt[n]{|E(X^n)|}\,/n = \limsup_n e\, \sqrt[n]{E(|X|^n)}\,/n. \end{aligned} \tag{21}$$

Hint: According to Stirling's formula $n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}$ as $n \to \infty$.]◇

Exercise 8. Suppose $X \sim \text{Gamma}(r, 1)$. Compute the moments of X by direct integration and use (15) to compute $E(e^{tX})$ for t's within the interval of convergence of the series. •

Let X be a random variable with MGF $M(t)$. The function K defined by $K(t) = \log\big(M(t)\big)$ is called the ***cumulant generating function*** of X; it's derivatives at 0, namely

$$\kappa_r := K^{(r)}(0), \quad r = 0,\, 1,\, 2,\, \dots\,, \tag{22}$$

are called the ***cumulants*** of X (or of M); the cumulants exist if M is finite in an open interval containing $t = 0$.

Exercise 9. Let X be a random variable having cumulants κ_0, κ_1, κ_2, … . Show that

$$\kappa_0 = 0,\ \kappa_1 = E(X), \text{ and } \kappa_2 = \text{Var}(X). \tag{23}$$ ◇

The exercises below explore exponential families, a topic of import throughout statistical theory. The exercises are wordy, but except for Exercises 15 and 16, and the last parts of Exercises 11, 17, and 18, the answers should be almost immediate.

The following setup is used throughout exercises 10–18. Let f be a piecewise continuous density on the real line whose MGF

$$M(t) = \int e^{tx} f(x)\, dx$$

is finite for all t's in some nonempty open interval. Denote the largest such interval by (t_l, t_r), and note that $t_l \le 0$ and $t_r \ge 0$. (t_l and t_r may be infinite.) Let $K(t) = \log M(t)$ be the corresponding cumulant generating function. Construct a one-parameter family $\{f_\theta\}_{t_r<\theta<t_r}$ of densities related to f by ***exponential tilting***, as follows: for each $\theta \in (t_l, t_r)$, set

$$f_\theta(x) = e^{\theta x} f(x)/M(\theta) = e^{\theta x - K(\theta)} f(x). \tag{24}$$

Exercise 10. Show that: (a) f_θ is a density; (b) f_θ has MGF $M_\theta(t) = M(t+\theta)/M(\theta)$, finite for $t \in (t_l - \theta, t_r - \theta)$. ◇

Exercise 11. (a) Let $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$ for x real. What are M, t_l, and t_r? Name the density corresponding to f_θ. (b) Repeat (a) with $f(x) = \exp(-x)$ for x nonnegative. (c) Repeat (a) with $f(x) = 1/\big(\pi \cosh(x)\big)$ for x real. [Hint for (c): $\cosh(x) := (e^{-x} + e^x)/2$. There is no common name for f_θ. Show that f_θ is the density of $\frac{1}{2}\log(F)$, where F has an unnormalized F distribution; give the degrees of freedom.] ◇

Exercise 12. Show that (in general) the cumulant generating function K_θ of f_θ satisfies

$$K_\theta(t) = K(t+\theta) - K(\theta), \tag{25_1}$$

$$\frac{d}{dt} K_\theta(t) = K'(t+\theta), \tag{25_2}$$

$$\frac{d^r}{dt^r} K_\theta(t) = K^{(r)}(t+\theta), \tag{25_3}$$

K' and $K^{(r)}$ denoting derivatives of the function K. (In these formulas, $t \in (t_l - \theta, t_r - \theta)$.) ⋄

Exercise 13. Denote the r^{th} cumulant of f_θ by $\kappa_r(\theta)$ (so $\kappa_r(\theta) = K_\theta^{(r)}(0)$). Show that

$$\kappa_1(\theta) = K'(\theta), \quad \kappa_2(\theta) = K''(\theta), \tag{26_1}$$

$$\kappa_r(\theta) = K^{(r)}(\theta) = \frac{d^j}{d\theta^j}\kappa_{r-j}(\theta), \tag{26_2}$$

the last equation holding for any $r \geq 1$ and $0 \leq j < r$. ⋄

Exercise 14. Show that $\kappa_2(\theta) > 0$ and deduce that on the interval (t_l, t_r), K' is strictly increasing and K is strictly convex. [Hint: Use the result of Exercise 9.] ⋄

Exercise 15. Let

$$a = \inf\{\, x : f(x) > 0 \,\} \quad \text{and} \quad b = \sup\{\, x : f(x) > 0 \,\}$$

be respectively the smallest and largest points of support of f. Show that

$$\kappa_1(t_l+) := \lim_{\theta \downarrow t_1} \kappa_1(\theta) = a \tag{27_1}$$

provided

$$t_l = -\infty \quad \text{or} \quad t_l > -\infty \text{ and } M(t_l+) = \infty;$$

moreover

$$\kappa_1(t_r-) := \lim_{\theta \uparrow t_r} \kappa_1(\theta) = b \tag{27_2}$$

provided

$$t_r = \infty \quad \text{or} \quad t_r < \infty \text{ and } M(t_r-) = \infty.$$

[Hints: You need only prove (27_2), because the argument for (27_1) is similar. In the case that $t_r < \infty$, first argue that $b = \infty$. In the case that $t_r = \infty$, first argue that if $f(x) > 0$ for all x in some nonempty interval (α, β), then there exists a finite number c such that $K(t) \geq c + \alpha t$ for all $t \geq 0$.] ⋄

Exercise 16. Show by example that (27) need not hold. ⋄

Exercise 17. Suppose (27) holds. Show that for each $\xi \in (a, b)$, the equation

$$\kappa_1(\theta) = \xi \tag{28}$$

has a unique root $\theta \in (t_l, t_r)$, and hence that corresponding to each such ξ there is a unique density in the family $\{f_\theta\}$ that has mean ξ. (This result is of central importance to the saddlepoint approximations to which we shall return.) Solve (28) in closed form for the densities in Exercise 11. [Hint: For the third density in Exercise 11, you may find the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ helpful. The formula is valid for $0 < z < 1$; you do not have to prove it.] ⋄

Exercise 18. Everything in the preceding discussion applies as well to discrete distributions that are not concentrated on a single point, with the understanding that f is now to be interpretated as a frequency function (aka probability mass function). Let $f = b(n, 1/2)$ be the Binomial frequency function for n trials and success probability $1/2$. Find M, t_l, and t_r, and identify the family $\{f_\theta\}$ as the binomial family $b(n, p)$, $0 < p < 1$. How does the so-called "canonical parameter" θ correspond to the so-called "natural parameter" p? Find $dp/d\theta$ as a function of p. To avoid notational confusion, let $\kappa_{r,p} = \kappa_r(\theta)$ refer to the cumulants of the binomial, considered as a function of p and θ, respectively. From the general formula (26_2) on cumulant recursion and the chain rule, derive the recurrence relation

$$\kappa_{r,p} = pq\frac{d}{dp}\kappa_{r-1,p}. \tag{29}$$

Use (29) and Maple to find the first 8 binomial cumulants. ⋄