

GENERALIZED MATRIX NEARNESS PROBLEMS*

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Abstract. We show that the global minimum solution of $\|A - BXC\|$ can be found in closed form with singular value decompositions and generalized singular value decompositions for a variety of constraints on X involving rank, norm, symmetry, two-sided product, and prescribed eigenvalue. This extends the solution of Friedland–Torokhti for the generalized rank-constrained approximation problem to other constraints and provides an alternative solution for rank constraint in terms of singular value decompositions. For more complicated constraints on X involving structures such as Toeplitz, Hankel, circulant, nonnegativity, stochasticity, positive semidefiniteness, prescribed eigenvector, etc., we prove that a simple iterative method is linearly and globally convergent to the global minimum solution.

Key words. matrix approximation, matrix nearness, structured matrices

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1. Introduction. In [10], Friedland and Torokhti found a closed-form analytic solution for the generalized rank-constrained matrix approximation problem

$$(1.1) \quad \min_{\text{rank}(X) \leq r} \|A - BXC\|$$

with $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, and $\|\cdot\|$ the Frobenius norm, generalizing the celebrated result of Eckart and Young [9]. We extend this work in several ways, replacing the rank constraint $\text{rank}(X) \leq r$ by

- ① norm constraint: $\|X\| \leq \rho$ for a given $\rho > 0$;
- ② two-sided product constraint: $FXG = H$ for given matrices F, G, H ;
- ③ spectral constraints: X has a prescribed eigenvalue λ or a prescribed eigenvector $v \neq 0$;
- ④ symmetry constraints: X is symmetric or skew-symmetric;
- ⑤ structure constraints: X is Toeplitz, Hankel, or circulant;
- ⑥ positivity constraints: X is positive semidefinite, correlation, nonnegative, stochastic, or doubly stochastic.

Note that ② includes the important special case $Xg = h$ for given vectors g, h . We shall provide closed-form analytic solutions for ①–④, using singular value decomposition for ①–③ and generalized singular value decomposition for ④, with the exception of the prescribed eigenvector problem—for this and for ⑤ and ⑥, we prove that an iterative algorithm, when applied to these problems, is

- (i) globally convergent, i.e., converges for any initial point;
- (ii) linearly convergent, i.e., error decreases exponentially to zero;

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- (iii) provably convergent, i.e., converges to a global minimizer, not just a stationary point or local minimizer.

As an addendum, we provide a simpler alternative solution to (1.1) in terms of singular value decompositions. While it is analytically equivalent to the solution of [10] in terms of projection matrices and pseudoinverses, which may be numerically unstable to compute, an obvious advantage of our solution is that it is stably computable via singular value decompositions.

We emphasize that we do not treat the problems ①–⑥ as constrained optimization problems and then apply general purpose nonlinear or convex optimization methods. While these problems are stated as optimization problems, our solutions are firmly rooted in numerical linear algebra, in the tradition of [11, 12, 15, 16, 17, 19, 22, 30, 35], and crucially rely on the matrix structures in these problems. In particular, none of our methods would involve taking derivatives; all of them are zeroth-order methods from the perspective of optimization.

Throughout this article, we assume that the dimensions of the matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$ satisfy $m \geq p$ and $n \geq q$ since otherwise we may simply add rows of zeros to A and B or columns of zeros to A and C . While all results are stated over \mathbb{R} , it is routine to extend them to \mathbb{C} .

2. Closed-form solutions via singular value decomposition. We simplify the objective function $\|A - BXC\|$ via singular value decomposition and the orthogonal invariance of Frobenius norm and matrix rank. Consider singular value decompositions

$$B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T, \quad C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^T$$

with $\Sigma_B = \text{diag}(\sigma_1(B), \dots, \sigma_s(B))$, $\Sigma_C = \text{diag}(\sigma_1(C), \dots, \sigma_t(C))$, $\text{rank}(B) = s$, $\text{rank}(C) = t$. Since the Frobenius norm is invariant under left and right multiplications by orthogonal matrices, we have

$$\|A - BXC\| = \left\| U_B^T A V_C - \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T X U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} \right\|.$$

Partition the respective matrices as

$$U_B^T A V_C = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad V_B^T X U_C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $X_{11}, A_{11} \in \mathbb{R}^{s \times t}$. We obtain

$$\begin{aligned} \|A - BXC\|^2 &= \left\| \begin{bmatrix} A_{11} - \Sigma_B X_{11} \Sigma_C & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\|^2 \\ (2.1) \quad &= \|A_{11} - \Sigma_B X_{11} \Sigma_C\|^2 + \|A_{12}\|^2 + \|A_{21}\|^2 + \|A_{22}\|^2. \end{aligned}$$

Note in particular that X_{12} , X_{21} , X_{22} do not appear in the final expression and that we are free to choose them within whatever constraint we impose on X .

2.1. Generalized rank-constrained approximation. As an illustration, we consider the Friedland–Torokhti rank approximation problem [10]

$$\min_{\text{rank}(X) \leq r} \|A - BXC\|.$$

Algorithm 1 Generalized rank-constrained approximation

INPUT: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, $r \geq 0$;

1: compute singular value decompositions $B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T$ and $C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^T$;

2: compute A_{11} from

$$U_B^T A V_C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$$

3: compute singular value decomposition $A_{11} = U \Sigma V^T$;

OUTPUT:

$$X = V_B \begin{bmatrix} \Sigma_B^{-1} U_r \Sigma_r V_r^T \Sigma_C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_C^T.$$

We may set X_{12} , X_{21} , X_{22} to be zero matrices in (2.1), and since

$$\text{rank}(\Sigma_B X_{11} \Sigma_C) = \text{rank}(X_{11}) = \text{rank}(V_B^T X U_C) = \text{rank}(X),$$

we only need to solve

$$\min_{\text{rank}(\Sigma_B X_{11} \Sigma_C) \leq r} \|A_{11} - \Sigma_B X_{11} \Sigma_C\|.$$

It immediately follows from the Eckart–Young theorem that the solution is given by $X_{11} = \Sigma_B^{-1} U_r \Sigma_r V_r^T \Sigma_C^{-1}$, with singular value decomposition $A_{11} = U \Sigma V^T$ and $U_r \Sigma_r V_r^T$ the best rank- r approximation of A_{11} . In fact, given that we have set all free parameters in $V_B^T X U_C$ to zero, this actually gives the minimum-norm solution. We summarize our solution in Algorithm 1. Note that computing X using the closed-form expression given in [10] literally would likely yield a significantly less accurate result because the expression involves forming products of explicit pseudoinverses (see [18, section 14.1] and [31, pp. 252 and 289] for an explanation of why these should be avoided).

2.2. Generalized prescribed eigenvalue approximation. A consequence of our previous solution is the solution to the prescribed eigenvalue approximation problem in ③. This problem requires square matrices, so $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$. Let $\lambda(X)$ denote the spectrum of $X \in \mathbb{R}^{p \times p}$, and let λ be a given value. We want

$$\min_{\lambda \in \lambda(X)} \|A - BXC\|.$$

The special case where $B = C = I$ was famously discussed by Wilkinson in [34].

Since

$$\lambda \in \lambda(X) \iff \text{rank}(X - \lambda I) \leq p - 1,$$

we have

$$\begin{aligned} \min_{\lambda \in \lambda(X)} \|A - BXC\| &= \min_{\lambda \in \lambda(X)} \|A - \lambda BC - B(X - \lambda I)C\| \\ &= \min_{\text{rank}(Y) \leq p-1} \|\tilde{A} - BYC\|, \end{aligned}$$

where $\tilde{A} = A - \lambda BC$, and the problem reduces to the one in section 2.1.

2.3. Generalized norm-constrained approximation. The problem in ①,

$$\min_{\|X\| \leq \rho} \|A - BXC\|,$$

is of course a special case of a norm-constrained least squares problem if we ignore the fact that the variables x_{ij} 's come from a matrix. It may thus be solved using general techniques in [1, section 5.3]. The advantage of our approach is that it preserves the matrix structure of the problem so that, for example, we just need to decompose matrices B and C instead of the matrix $C \otimes B$, which is an order of magnitude larger.

By (2.1), we may set X_{12} , X_{21} , X_{22} to be zero matrices. Then

$$\|X_{11}\| = \|X\| \leq \rho.$$

If $\|\Sigma_B^{-1}A_{11}\Sigma_C^{-1}\| \leq \rho$, then the solution is simply $X_{11} = \Sigma_B^{-1}A_{11}\Sigma_C^{-1}$. So we may suppose that $\|\Sigma_B^{-1}A_{11}\Sigma_C^{-1}\| > \rho$, in which case the solution must lie on the boundary, i.e., $\|X_{11}\| = \rho$. To see this, note that if $\|X_{11}\| < \rho$, then $X_{11} \neq \Sigma_B^{-1}A_{11}\Sigma_C^{-1}$ since $\|\Sigma_B^{-1}A_{11}\Sigma_C^{-1}\| > \rho$. Hence, there exists some $t \in (0, 1)$ so that $\|t\Sigma_B^{-1}A_{11}\Sigma_C^{-1} + (1-t)X_{11}\| = \rho$ and

$$\begin{aligned} \|A_{11} - \Sigma_B[t\Sigma_B^{-1}A_{11}\Sigma_C^{-1} + (1-t)X_{11}]\Sigma_C\| &= (1-t)\|A_{11} - \Sigma_B X_{11}\Sigma_C\| \\ &< \|A_{11} - \Sigma_B X_{11}\Sigma_C\|, \end{aligned}$$

contradicting the minimality of X_{11} .

So the inequality constraint may be replaced by an equality constraint $\|X_{11}\| = \rho$, and standard Lagrange multiplier theory [29, Chapter 14] applied to

$$\|A_{11} - \Sigma_B X_{11}\Sigma_C\|^2 + \lambda(\|X_{11}\|^2 - \rho^2)$$

gives

$$(2.2) \quad \begin{cases} \Sigma_B(\Sigma_B X_{11}\Sigma_C - A_{11})\Sigma_C + \lambda X_{11} = 0, \\ \|X_{11}\|^2 = \rho^2. \end{cases}$$

If (X_1, λ_1) and (X_2, λ_2) are both solutions to (2.2), then

$$(2.3) \quad \begin{aligned} \Sigma_B(\Sigma_B X_1\Sigma_C - A_{11})\Sigma_C + \lambda_1 X_1 &= 0, \\ \Sigma_B(\Sigma_B X_2\Sigma_C - A_{11})\Sigma_C + \lambda_2 X_2 &= 0, \end{aligned}$$

and $\|X_1\|^2 = \|X_2\|^2 = \rho^2$. Left multiply the first equation in (2.3) by X_1^\top and the second by X_2^\top , then take the trace and subtract. We get

$$(2.4) \quad \begin{aligned} \|\Sigma_B X_2\Sigma_C\|^2 - \|\Sigma_B X_1\Sigma_C\|^2 - (\text{tr}(X_2^\top \Sigma_B A_{11}\Sigma_C) - \text{tr}(X_1^\top \Sigma_B A_{11}\Sigma_C)) \\ = \lambda_1 \|X_1\|^2 - \lambda_2 \|X_2\|^2. \end{aligned}$$

Left multiply the first equation in (2.3) by X_2^\top and the second by X_1^\top , then take trace and subtract. We get

$$(2.5) \quad -(\text{tr}(X_2^\top \Sigma_B A_{11}\Sigma_C) - \text{tr}(X_1^\top \Sigma_B A_{11}\Sigma_C)) = -(\lambda_1 - \lambda_2) \text{tr}(X_1^\top X_2).$$

Adding (2.4) and (2.5) and noting that $\|X_1\|^2 = \|X_2\|^2$, we get

$$\|A_{11} - \Sigma_B X_2\Sigma_C\| - \|A_{11} - \Sigma_B X_1\Sigma_C\| = \frac{\lambda_1 - \lambda_2}{2} \|X_1 - X_2\|^2.$$

Hence,

$$\lambda_1 > \lambda_2 \quad \Rightarrow \quad \|A_{11} - \Sigma_B X_2\Sigma_C\| > \|A_{11} - \Sigma_B X_1\Sigma_C\|.$$

Algorithm 2 Generalized norm-constrained approximation

INPUT: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, $\rho > 0$;

- 1: compute singular value decompositions $B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T$ and $C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^T$;
- 2: compute A_{11} from

$$U_B^T A V_C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$$

- 3: set $\sigma_{ij} = \sigma_i(B)\sigma_j(C)$;
- 4: calculate largest root λ of $f(\lambda) = \rho^2$;
- 5: set X_{11} with entries

$$x_{ij} = \frac{a_{ij}}{\sigma_{ij} + \lambda\sigma_{ij}^{-1}};$$

OUTPUT:

$$X = V_B \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} U_C^T.$$

The smallest $\|A_{11} - \Sigma_B X_{11} \Sigma_C\|$ corresponds to the largest λ . Thus, we seek the solution (X_{11}, λ) to (2.2) with the largest λ . Let $A_{11} = (a_{ij})$, $X_{11} = (x_{ij})$, $\sigma_{ij} = \sigma_i(B)\sigma_j(C)$. The solution to the first equation (2.2) is

$$x_{ij} = \frac{a_{ij}}{\sigma_{ij} + \lambda\sigma_{ij}^{-1}}.$$

Plugging into the second equation in (2.2), we obtain the *secular equation*

$$(2.6) \quad f(\lambda) := \sum_{i=1}^s \sum_{j=1}^t \frac{a_{ij}^2}{(\sigma_{ij} + \lambda\sigma_{ij}^{-1})^2} = \rho^2.$$

The secular equation is a ubiquitous univariate nonlinear equation in matrix computation; see [13, section 12.1.1] or [6, section 5.3.3] for a discussion of its properties. In particular, f has poles at $-\sigma_{ij}^2$ and $\lim_{\lambda \rightarrow \pm\infty} f(\lambda) = 0$, so $f(\lambda) = \rho^2$ only has a finite number of solutions. We may find all real roots of $f(\lambda) = \rho^2$ using any standard univariate root finder, e.g., Newton–Raphson, regula falsi, Brent, etc., and identify the largest root. A well-known trick is to instead apply the root finder to $1/f(\lambda) = 1/\rho^2$, as $1/f$ is close to linear in the vicinity of a root and convergence will be extremely fast. We summarize this solution in Algorithm 2.

2.4. Generalized two-sided product-constrained approximation. Here we consider the problem ②:

$$(2.7) \quad \min_{FXG=H} \|A - BXC\|.$$

This may be viewed as a least squares counterpart to various *two-sided linear matrix equations* such as

$$\begin{cases} BXC = A, \\ FXG = H, \end{cases} \quad \text{or} \quad BXC + FXG = H,$$

which have been studied in [3, 26, 33].

There is no loss of generality in assuming that F has full row rank and G has full column rank. Otherwise, we may simply take the reduced QR factorizations $F = Q_F R_F$, where $Q_F^T Q_F = I$ and $G^T = Q_G R_G$, where $Q_G^T Q_G = I$; observe that $F X G = H$ then becomes $R_F X R_G^T = Q_F^T H Q_G$, i.e., of the form $F' X G' = H'$, where $F' := R_F$ has full row rank, $G' := R_G^T$ has full column rank, and $H' := Q_F^T H Q_G$.

For a closed-form solution, we will also need to assume that B has full column rank and C has full row rank, i.e., $\text{rank}(B) = p$, $\text{rank}(C) = q$. Unlike the case of F and G , there is a loss of generality in imposing these conditions on B and C . However, the case of rank-deficient B and C could be easily solved with our iterative algorithm in section 4.

We start with the simpler version

$$(2.8) \quad \min_{F X G = H} \|A - X\|.$$

We claim that the solution is given by $\widehat{X} = A + F^T(F F^T)^{-1}(H - F A G)(G^T G)^{-1} G^T$. First, observe that

$$F \widehat{X} G = F [A + F^T(F F^T)^{-1}(H - F A G)(G^T G)^{-1} G^T] G = F A G + H - F A G = H.$$

Next, recall that if $\|\cdot\|_2$ denotes the spectral norm (matrix 2-norm), then $\|Y Z\| \leq \|Y\|_2 \|Z\|$ for any $Y \in \mathbb{R}^{m \times n}$, $Z = [z_1, \dots, z_p] \in \mathbb{R}^{n \times p}$ as

$$\|Y Z\|^2 = \sum_{i=1}^p \|Y z_i\|_2^2 \leq \sum_{i=1}^p \|Y\|_2^2 \|z_i\|_2^2 = \|Y\|_2^2 \left[\sum_{i=1}^p \|z_i\|_2^2 \right] = \|Y\|_2^2 \|Z\|^2.$$

Thus, for any X satisfying $F X G = H$, we have

$$\begin{aligned} \|A - \widehat{X}\| &= \|F^T(F F^T)^{-1}(H - F A G)(G^T G)^{-1} G^T\| \\ &= \|F^T(F F^T)^{-1} F(A - X)G(G^T G)^{-1} G^T\| \\ &\leq \|F^T(F F^T)^{-1} F\|_2 \|A - X\| \|G(G^T G)^{-1} G^T\|_2 = \|A - X\| \end{aligned}$$

since $F^T(F F^T)^{-1} F$ and $G(G^T G)^{-1} G^T$ are orthogonal projectors.

Now for the generalized problem (2.7). Following our notations at the beginning of section 2, given that we have assumed $\text{rank}(B) = p$ and $\text{rank}(C) = q$, we have

$$V_B^T X U_C = X_{11}, \quad U_B^T A V_C = A_{11}.$$

Let $F_B := F V_B \Sigma_B^{-1}$ and $G_C := \Sigma_C^{-1} U_C^T G$. The constraint $F X G = H$ is then equivalent to

$$F_B (\Sigma_B V_B^T X U_C \Sigma_C) G_C = H.$$

By the solution to (2.8), we get

$$X = V_B \Sigma_B^{-1} [U_B^T A V_C + F_B^T (F_B F_B^T)^{-1} (H - F_B U_B^T A V_C G_C) (G_C^T G_C)^{-1} G_C^T] \Sigma_C^{-1} U_C^T.$$

We summarize this solution in Algorithm 3.

Algorithm 3 Generalized two-sided product-constrained approximation

INPUT: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, $H \in \mathbb{R}^{k \times l}$, $F \in \mathbb{R}^{k \times p}$, $G \in \mathbb{R}^{q \times l}$;
 1: compute singular value decompositions $B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T$ and $C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^T$;
 2: compute $F_B = FV_B \Sigma_B^{-1}$ and $G_C = \Sigma_C^{-1} U_C^T G$;
 OUTPUT:

$$X = V_B \Sigma_B^{-1} [U_B^T A V_C + F_B^T (F_B F_B^T)^{-1} (H - F_B U_B^T A V_C G_C) (G_C^T G_C)^{-1} G_C^T] \Sigma_C^{-1} U_C^T.$$

3. Closed-form solutions via generalized singular value decomposition.

We now address ④, starting with symmetry constraint

$$(3.1) \quad \min_{X=X^T} \|A - BXC\|$$

and deferring skew-symmetry constraint to later. Since X is necessarily a square matrix, we require the number of columns in B to equal the number of rows in C . So let $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times p}$. The special case where $C = I$ is called the symmetric Procrustes problem [2, 24] and was solved in [7, 16]. Exact versions of this problem, i.e., seeking symmetric solutions to $BXC = A$, have been studied in [4, 23]. The special case $B = C = I$ is elementary and well known: The projection of A onto the spaces of symmetric and skew-symmetric matrices are given by the additive decomposition $A = (A + A^T)/2 + (A - A^T)/2$ into two orthogonal components.

To preserve the symmetry, the singular value decomposition is not useful, as $V_B^T X U_C$ is generally not symmetric for a symmetric X . However, the generalized singular value decomposition [27] is perfect for our purpose. We remind the reader of this result.

THEOREM 3.1 (Paige–Saunders). *Let $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times p}$, $k = \text{rank}(\begin{bmatrix} B \\ C^T \end{bmatrix})$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{p \times p}$, $W \in \mathbb{R}^{k \times k}$, and $Q \in \mathbb{R}^{n \times n}$ with*

$$(3.2) \quad U^T B Q = \Sigma_B \begin{bmatrix} W^T R & 0 \end{bmatrix}, \quad V^T C^T Q = \Sigma_C \begin{bmatrix} W^T R & 0 \end{bmatrix},$$

where

$$\Sigma_B = \begin{bmatrix} I_B & & \\ & S_B & \\ & & O_B \end{bmatrix}, \quad \Sigma_C = \begin{bmatrix} O_C & & \\ & S_C & \\ & & I_C \end{bmatrix},$$

and $R \in \mathbb{R}^{k \times k}$ is nonsingular with singular values equal to the nonzero singular values of $\begin{bmatrix} B \\ C^T \end{bmatrix}$. Here $I_B \in \mathbb{R}^{r \times r}$ and $I_C \in \mathbb{R}^{k-r-s \times k-r-s}$ are identity matrices, $O_B \in \mathbb{R}^{(m-r-s) \times (k-r-s)}$ and $O_C \in \mathbb{R}^{(p-k+r) \times r}$ are zero matrices with possibly no rows or columns, and $S_B = \text{diag}(\beta_{r+1}, \dots, \beta_{r+s})$, $S_C = \text{diag}(\gamma_{r+1}, \dots, \gamma_{r+s}) \in \mathbb{R}^{s \times s}$ with

$$1 > \beta_{r+1} \geq \dots \geq \beta_{r+s} > 0, \quad 0 < \gamma_{r+1} \leq \dots \leq \gamma_{r+s} < 1.$$

Following Theorem 3.1, let

$$M = Q \begin{bmatrix} R^{-1} W & 0 \\ 0 & I \end{bmatrix}.$$

Then (3.2) becomes

$$U^T B M = [\Sigma_B \ 0], \quad V^T C^T M = [\Sigma_C \ 0].$$

Then

$$\begin{aligned} \|A - BXC\| &= \|A - U [\Sigma_B \ 0] M^{-1} X M^{-T} [\Sigma_C \ 0]^T V^T\| \\ &= \left\| U^T A V - [\Sigma_B \ 0] M^{-1} X M^{-T} \begin{bmatrix} \Sigma_C^T \\ 0 \end{bmatrix} \right\|. \end{aligned}$$

Partition the respective matrices to conform to the block structure of Σ_B and Σ_C^T :

$$U^T A V = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad M^{-1} X M^{-T} = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}.$$

We get

$$U^T A V - [\Sigma_B \ 0] M^{-1} X M^{-T} \begin{bmatrix} \Sigma_C^T \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} - X_{12} S_C & A_{13} - X_{13} \\ A_{21} & A_{22} - S_B X_{22} S_C & A_{23} - S_B X_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

and (3.1) reduces to minimizing

$$\|A_{12} - X_{12} S_C\|^2 + \|A_{13} - X_{13}\|^2 + \|A_{23} - S_B X_{23}\|^2 + \|A_{22} - S_B X_{22} S_C\|^2.$$

Since $M^{-1} X M^{-T}$ is symmetric, the solution is easily seen to be

$$\begin{aligned} X_{12} &= A_{12} S_C^{-1}, & X_{13} &= A_{13}, & X_{23} &= S_B^{-1} A_{23}, \\ X_{21} &= S_C^{-T} A_{12}^T, & X_{31} &= A_{13}^T, & X_{32} &= A_{23}^T S_B^{-T}; \end{aligned}$$

$X_{22} = (x_{ij}) \in \mathbb{R}^{s \times s}$ is given by defining $\sigma_{ij} := \beta_{r+i} \gamma_{r+j}$ and setting

$$x_{ij} = x_{ji} = \begin{cases} \frac{a_{ij} \sigma_{ij} + a_{ji} \sigma_{ji}}{\sigma_{ij}^2 + \sigma_{ji}^2}, & \sigma_{ij}^2 + \sigma_{ji}^2 \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \dots, s$. The other blocks $X_{11}, X_{33}, X_{44}, X_{14} = X_{41}^T, X_{24} = X_{42}^T, X_{34} = X_{43}^T$ are all set to be zero matrices. Note that although we have set the free parameters in X_{22} to be zeros, they may be arbitrary as long as $X_{22}^T = X_{22}$. We summarize our solution in Algorithm 4.

It is easy to adapt the solution above for skew-symmetric matrices, the only change being that since $M^{-1} X M^{-T}$ is now skew-symmetric, we want

$$\begin{aligned} X_{12} &= A_{12} S_C^{-1}, & X_{13} &= A_{13}, & X_{23} &= S_B^{-1} A_{23}, \\ X_{21} &= -S_C^{-T} A_{12}^T, & X_{31} &= -A_{13}^T, & X_{32} &= -A_{23}^T S_B^{-T}; \end{aligned}$$

$X_{22} = (x_{ij}) \in \mathbb{R}^{s \times s}$ is given by defining $\sigma_{ij} := \beta_{r+i} \gamma_{r+j}$ and setting

$$x_{ij} = -x_{ji} = \begin{cases} \frac{a_{ij} \sigma_{ij} - a_{ji} \sigma_{ji}}{\sigma_{ij}^2 + \sigma_{ji}^2} & \sigma_{ij}^2 + \sigma_{ji}^2 \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \dots, s$.

Algorithm 4 Generalized symmetry-constrained approximation

INPUT: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, $k \geq 0$;

1: compute generalized singular value decomposition

$$U^T B Q = \Sigma_B [W^T R \ 0], \quad V^T C^T Q = \Sigma_C [W^T R \ 0]$$

with $\Sigma_B = \text{diag}(I_B, S_B, O_B)$, $\Sigma_C = \text{diag}(O_C, S_C, I_C)$;

2: compute

$$M = Q \begin{bmatrix} R^{-1} W & 0 \\ 0 & I \end{bmatrix};$$

3: compute

$$U^T A V = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix};$$

4: compute $X_{22} = (x_{ij})$ as

$$x_{ij} = x_{ji} = \begin{cases} \frac{a_{ij}\sigma_{ij} + a_{ji}\sigma_{ji}}{\sigma_{ij}^2 + \sigma_{ji}^2}, & \sigma_{ij}^2 + \sigma_{ji}^2 \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

OUTPUT:

$$\hat{X} = M \begin{bmatrix} 0 & A_{12} S_C^{-1} & A_{13} & 0 \\ S_C^{-T} A_{12}^T & X_{22} & S_B^{-1} A_{23} & 0 \\ A_{13}^T & A_{23}^T S_B^{-T} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} M^T.$$

4. Iterative solution. The other problems ⑤ and ⑥ and the prescribed eigenvector problem take the form

$$(4.1) \quad \min_{X \in \mathcal{S}} \|A - BXC\|,$$

where \mathcal{S} is a closed convex set of matrices having the requisite property. Although we are unable to obtain a closed-form solution for these problems directly, we may solve them by alternating between two subproblems with closed-form solutions:

- (a) for any given A, B, C, Y , and ρ , we have a closed-form solution for the norm-constrained problem

$$\min_{\|X - Y\| \leq \rho} \|A - BXC\|;$$

- (b) for any A , when $B = I$ and $C = I$, we have a closed-form solution for the special case

$$\min_{Y \in \mathcal{S}} \|A - Y\|.$$

The problem (a) is a minor variant of the problem ① solved in section 2.3, with the ρ -ball centered at Y instead of 0. The problem (b) is a projection onto the set of interest \mathcal{S} , which we will solve in section 5 for ⑤, ⑥, and the prescribed eigenvector problem. Sometimes we will have to project twice to different sets \mathcal{S}_1 and \mathcal{S}_2 ; this

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happens when there are closed-form expressions for projections onto \mathcal{S}_1 and \mathcal{S}_2 but none for the set of interest $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$. An example is when \mathcal{S} is the set of correlation matrices, with \mathcal{S}_1 the set of positive semidefinite matrices and \mathcal{S}_2 the set of matrices with ones on the diagonal [19]. We will discuss this variant in section 4.2.

Alternating between (a) and (b) trades off between minimizing $\|A - BXC\|$ and staying close to within distance ρ of the projection $Y \in \mathcal{S}$. However, if we simply alternate between these two subproblems, the iterates may end up simply oscillating between two fixed points. The well-known solution is to introduce a *Dijkstra correction* Z [8] so that we have

$$(4.2) \quad Y_{k+1} = \operatorname{argmin} \{\|X_k - Z_k - Y\| : Y \in \mathcal{S}\},$$

$$(4.3) \quad X_{k+1} = \operatorname{argmin} \{\|A - BXC\| : \|X - Z_k - Y_{k+1}\| \leq \rho_k, X \in \mathbb{R}^{p \times q}\},$$

$$(4.4) \quad Z_{k+1} = Z_k - X_{k+1} + Y_{k+1}.$$

In Theorems 4.3 and 4.4, we prove the linear and global convergence of this iterative algorithm to a global minimizer of (4.1).

We will see in section 5 that (4.2) is readily solvable for ⑤, ⑥, and the prescribed eigenvector problem. If we set $W = Z_k + Y_{k+1}$, $X' = X - W$, $A' = A - BWC$, then step (4.3) becomes

$$\min_{\|X'\| \leq \rho_k} \|A' - BX'C\|;$$

i.e., it is exactly the norm-constrained problem ① that we solved in section 2.3. For the sequence of ρ_k , let $a_{ij}(W)$ denote the (i, j) th entry of $A' = A - BWC$. As we discussed in section 2.3, for any fixed W , there is a bijection between $\rho > 0$ and the largest root $\lambda > 0$ of the secular equation

$$(4.5) \quad f(\lambda, W) := \sum_{i=1}^s \sum_{j=1}^t \frac{a_{ij}(W)^2}{(\sigma_{ij} + \lambda \sigma_{ij}^{-1})^2} = \rho^2.$$

We will show in Theorem 4.4 that when B has full column rank and C has full row rank, choosing

$$(4.6) \quad \lambda = \sigma_{\min}(B)\sigma_{\min}(C)\sigma_{\max}(B)\sigma_{\max}(C)$$

gives us the fastest rate of convergence. In this case, we have

$$\rho(W) = \sqrt{f(\sigma_{\min}(B)\sigma_{\min}(C)\sigma_{\max}(B)\sigma_{\max}(C), W)}.$$

Note that we set λ to be a fixed constant but that ρ generally depends on W . We write $\rho(W)$ to emphasize this dependence; in (4.3), $\rho_k = \rho(Z_k + Y_{k+1})$. Nevertheless, like Algorithm 2, ρ_k will not make an appearance in our iterative algorithm, which only requires λ . Unlike Algorithm 2, our iterative algorithm fixes a value of λ at the beginning, saving us the effort of solving a secular equation.

We summarize the above discussion in Algorithm 5. Aside from lines 4 and 10, the rest of the algorithm is identical to Algorithm 2 but sans the secular equation step. Yet another advantage is that the optimal λ in (4.6) is available to us “for free” since the algorithm requires computing the singular value decompositions of B and C to solve (4.3).

Algorithm 5 Iterative algorithm for generalized nearness problems

- INPUT: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$;
 1: precompute $B = U_B \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T$ and $C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^T$, $\sigma_{ij} = \sigma_i(B)\sigma_j(C)$;
 2: set $\lambda = \sigma_{\min}(B)\sigma_{\min}(C)\sigma_{\max}(B)\sigma_{\max}(C)$;
 3: initialize $X_0, Y_0, Z_0, k = 0$;
 4: compute Y_{k+1} by projecting $X_k - Z_k$ to the desired set \mathcal{S} ;
 5: compute $W = Y_{k+1} + Z_k$;
 6: compute $A' = A - BWC$;
 7: compute A_{11} from

$$U_B^T A' V_C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$$

- 8: compute $X_{11} = (x_{ij})$ as

$$x_{ij} = \frac{a_{ij}}{\sigma_{ij} + \lambda \sigma_{ij}^{-1}};$$

- 9: compute

$$X_{k+1} = V_B \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} U_C^T + W;$$

- 10: compute $Z_{k+1} = Z_k - X_{k+1} + Y_{k+1}$;
 11: $k = k + 1$ and go to line 4.
-

4.1. Convergence theorems. We will now show that the iterates generated by Algorithm 5 always converge to the global solution of (4.1) for any initialization and that the convergence rate is linear. By (2.2), a solution X_{k+1} of (4.3) satisfies $B^T(BX_{k+1}C - A)C^T + \lambda(X_{k+1} - Z_k - Y_{k+1}) = 0$; plugging into (4.4), we get

$$(4.7) \quad \lambda Z_{k+1} = B^T(BX_{k+1}C - A)C^T.$$

A solution Y_{k+1} of (4.2) clearly satisfies

$$\begin{aligned} 0 &\leq \|X_k - Z_k - Y\|^2 - \|X_k - Z_k - Y_{k+1}\|^2 \\ &= 2 \operatorname{tr}((Y_{k+1} - X_k + Z_k)^T(Y - Y_{k+1})) + \|Y - Y_{k+1}\|^2 \end{aligned}$$

for all $Y \in \mathcal{S}$. Thus, $\operatorname{tr}((Y_{k+1} - X_k + Z_k)^T(Y - Y_{k+1})) \geq 0$ for all $Y \in \mathcal{S}$. Plugging into (4.4), we get

$$(4.8) \quad \lambda \operatorname{tr}((X_k - X_{k+1} - Z_{k+1})^T(Y_{k+1} - Y)) \geq 0$$

for all $Y \in \mathcal{S}$.

As a sanity check, we first establish that a global minimizer of $\|A - BXC\|^2$ is guaranteed to exist for various \mathcal{S} of interest to us. The one exception is the case when \mathcal{S} is the positive semidefinite cone, which we will discuss later. Recall that $m \geq p$ and $n \geq q$ as at the end of section 1.

PROPOSITION 4.1 (existence and characterization of global minimizer). *For any $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, the function $X \mapsto \|A - BXC\|^2$ attains its infimum on any \mathcal{S} defined by any of the constraints ①–⑥ with the exception of positive semidefinite case in ⑥. In addition, X_* is a global minimizer if and only if for*

$$(4.9) \quad \lambda Z_* := B^T(BX_*C - A)C^T,$$

we have

$$(4.10) \quad \lambda \operatorname{tr}(Z_*^\top(Y - X_*)) \geq 0$$

for all $Y \in \mathcal{S}$. Note that we introduced the parameter λ in (4.9) for consistency with (4.7).

Proof. This holds as long as the sublevel set $\{X \in \mathcal{S} : \|A - BXC\|^2 \leq \alpha\}$ is compact in $\mathbb{R}^{p \times q}$ for some $\alpha > 0$. Obviously, this trivially holds for compact \mathcal{S} . For closed and unbounded \mathcal{S} , it is true if $\|A - BXC\|^2 \rightarrow \infty$ whenever $\|X\| \rightarrow \infty$. If B and C are both full rank, then the last property is easy to check. But we do not assume this; in fact, we do not even exclude extreme cases like $B = 0$ or $C = 0$. Fortunately, the issue is easy to fix.

Consider the linear map $\Phi: \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{m \times n}$, $X \mapsto BXC$. Let $\ker(\Phi) = \{X \in \mathbb{R}^{m \times n} : BXC = 0\}$ be its null space, and let $\ker(\Phi)^\perp$ be the orthogonal complement under the trace inner product on $\mathbb{R}^{p \times q}$, i.e., $\langle X, Y \rangle = \operatorname{tr}(X^\top Y)$. Let $\pi: \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{p \times q}$ be the orthogonal projection onto $\ker(\Phi)^\perp$. Then $\|A - BXC\|^2 = \|A - B\pi(X)C\|^2$, and so we only need to consider the function $X \mapsto \|A - BXC\|^2$ restricted on $\pi(\mathcal{S}) \subseteq \ker(\Phi)^\perp$.

It remains to check that the function $X \mapsto \|A - BXC\|^2$ has compact sublevel sets when restricted to $\pi(\mathcal{S})$. Restricting X to be orthogonal to $\ker(\Phi)$ ensures that for unbounded \mathcal{S} , we have $\|A - BXC\|^2 \rightarrow \infty$ as $\|X\| \rightarrow \infty$; i.e., the sublevel sets $\{X \in \pi(\mathcal{S}) : \|A - BXC\|^2 \leq \alpha\}$ are all bounded and would be compact as long as $\pi(\mathcal{S})$ is closed. We check that it is indeed closed for each of ①–⑥ except the positive semidefinite case in ⑥:

- (i) If \mathcal{S} is compact, then $\pi(\mathcal{S})$ is closed since π is continuous. This covers ① and the correlation, stochastic, and doubly stochastic cases in ⑥.
- (ii) If \mathcal{S} is an affine subspace, then $\pi(\mathcal{S})$ is also an affine subspace. Any finite-dimensional affine subspace is closed. This covers ②, ④, ⑤, and the prescribed eigenvector case in ③.
- (iii) If \mathcal{S} is a polyhedral cone, then $\pi(\mathcal{S})$ is also a polyhedral cone. Every polyhedral cone is closed because it is a finite intersection of halfspaces. This covers the nonnegative case in ⑥.

This leaves the prescribed eigenvalue case in ③, which, as we saw in section 2.2, is a special case of (1.1) and has a closed-form solution, and thus existence of its global minimizer is guaranteed.

For any $Y \in \mathcal{S}$, since X_* is a global minimizer, we must have

$$(4.11) \quad \begin{aligned} 0 &\leq \|A - BYC\|^2 - \|A - BX_*C\|^2 \\ &= 2 \operatorname{tr}((B^\top(BX_*C - A)C^\top)^\top(Y - X_*)) + \|B(Y - X_*)C\|^2 \end{aligned}$$

and thus

$$\operatorname{tr}((B^\top(BX_*C - A)C^\top)^\top(Y - X_*)) \geq 0.$$

Let Z_* be defined by (4.9). Then the last inequality becomes (4.10). Conversely, if X_* is such that (4.10) holds for all $Y \in \mathcal{S}$, then by (4.11), we have $\|A - BX_*C\|^2 \leq \|A - BYC\|^2$ for all $Y \in \mathcal{S}$, and so X_* is a global minimizer. \square

That Proposition 4.1 may not hold when \mathcal{S} is the positive semidefinite cone deserves a word. For one, it is a bit of a surprise since for the special case $B = C = I$, the nearest positive semidefinite matrix problem has a well-known solution given by the

polar decomposition (cf. section 5). To see that the existence of a global minimizer is not guaranteed for more general B and C , consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = I$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Write $X = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Then

$$\max_{X \geq 0} \|A - BXC\|^2 = \max_{a \geq 0, c \geq 0, ac \geq b^2} (b - 1)^2 + c^2.$$

Clearly, the objective on the right can be made arbitrarily close to zero by taking, say, $b = 1$, $a = b^2/c$, and $c \rightarrow 0$, but the objective cannot equal zero.

LEMMA 4.2. *The iterate (X_k, Z_k) of Algorithm 5 satisfies*

$$\begin{aligned} (4.12) \quad & \|X_k - X_*\|^2 + \|Z_k - Z_*\|^2 - \|X_{k+1} - X_*\|^2 - \|Z_{k+1} - Z_*\|^2 \\ & \geq \|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2 + \frac{2}{\lambda} \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_k - X_{k+1}\|^2 \\ & \quad + \frac{2}{\lambda} \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_{k+1} - X_*\|^2. \end{aligned}$$

Proof. It is easy to see that

$$(4.13) \quad \begin{aligned} \sigma_{\max}(B)^2 \sigma_{\max}(C)^2 \|X_1 - X_2\|^2 & \geq \text{tr}[(B^T B(X_1 - X_2) C C^T)^T (X_1 - X_2)], \\ \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_1 - X_2\|^2 & \leq \text{tr}[(B^T B(X_1 - X_2) C C^T)^T (X_1 - X_2)]. \end{aligned}$$

Combining (4.9), (4.13), and (4.7), we have

$$(4.14) \quad \lambda \text{tr}((X_{k+1} - X_*)^T (Z_{k+1} - Z_*)) \geq \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_{k+1} - X_*\|^2.$$

Combining (4.10) and (4.8), we have

$$(4.15) \quad \lambda \text{tr}((Y_{k+1} - X_*)(X_k - X_{k+1} - Z_{k+1} + Z_*)) \geq 0.$$

Adding (4.14) and (4.15) and applying (4.4), we have

$$(4.16) \quad \begin{aligned} \lambda \text{tr}[(Z_k - Z_{k+1})^T (Z_{k+1} - Z_* - (X_k - X_{k+1}))] \\ + \lambda \text{tr}((X_{k+1} - X_*)^T (X_k - X_{k+1})) \geq \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_{k+1} - X_*\|^2. \end{aligned}$$

Dividing (4.16) by λ and using the identity $\|a - c\|^2 - \|b - c\|^2 = 2 \text{tr}((a - c)^T (a - b)) - \|a - b\|^2$,

$$(4.17) \quad \begin{aligned} \|X_k - X_*\|^2 + \|Z_k - Z_*\|^2 - \|X_{k+1} - X_*\|^2 - \|Z_{k+1} - Z_*\|^2 & \geq \|X_k - X_{k+1}\|^2 \\ + \|Z_k - Z_{k+1}\|^2 + 2 \text{tr}((Z_k - Z_{k+1})^T (X_k - X_{k+1})) \\ + \frac{2}{\lambda} \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_{k+1} - X_*\|^2. \end{aligned}$$

From (4.13) and (4.7), we have

$$\lambda \text{tr}((Z_k - Z_{k+1})^T (X_k - X_{k+1})) \geq \sigma_{\min}(B)^2 \sigma_{\min}(C)^2 \|X_k - X_{k+1}\|^2.$$

Plug this into (4.17), and we obtain the desired result. □

THEOREM 4.3 (global convergence). *For any $\lambda > 0$, the iterate (X_k, Z_k) of Algorithm 5 converges to some (X_*, Z_*) satisfying (4.9) and (4.10).*

Proof. By (4.12), the sequence $(\|X_k - X_*\|^2 + \|Z_k - Z_*\|^2)_{k=0}^\infty$ is monotone decreasing and thus convergent. This also implies that $(X_k)_{k=0}^\infty$ and $(Z_k)_{k=0}^\infty$ are bounded and therefore have convergent subsequences $(X_{k_j})_{j=0}^\infty$ with limit \tilde{X} and $(Z_{k_j})_{j=0}^\infty$ with limit \tilde{Z} . Taking limits in (4.7) and (4.8) over these subsequences, we obtain

$$\lambda \tilde{Z} = B^\top(B\tilde{X}C - A)C^\top, \quad \lambda \operatorname{tr}(\tilde{Z}^\top(Y - \tilde{X})) \geq 0$$

for all $Y \in \mathcal{S}$. Thus, (\tilde{X}, \tilde{Z}) satisfies (4.9) and (4.10), implying that \tilde{X} is a global minimizer. Let $X_* = \tilde{X}$ and $Z_* = \tilde{Z}$. Since $(\|X_k - X_*\|^2 + \|Z_k - Z_*\|^2)_{k=0}^\infty$ is convergent, we must have $X_k \rightarrow X_*$ and $Z_k \rightarrow Z_*$. \square

THEOREM 4.4 (linear convergence). *Suppose B has full column rank and C has full row rank. Then*

$$\|X_k - X_*\|^2 + \|Z_k - Z_*\|^2 \geq (1 + \delta)(\|X_{k+1} - X_*\|^2 + \|Z_{k+1} - Z_*\|^2)$$

with

$$\delta = \left[\frac{\lambda}{2\sigma_{\min}(B)^2\sigma_{\min}(C)^2} + \frac{\sigma_{\max}(B)^2\sigma_{\max}(C)^2}{2\lambda} \right]^{-1}.$$

In particular, choosing

$$\lambda = \sigma_{\min}(B)\sigma_{\min}(C)\sigma_{\max}(B)\sigma_{\max}(C)$$

maximizes the convergence rate with

$$\delta = \frac{\sigma_{\min}(B)\sigma_{\min}(C)}{\sigma_{\max}(B)\sigma_{\max}(C)} = \frac{1}{\kappa_2(B)\kappa_2(C)}.$$

Proof. If B has full column rank and C has full row rank, then

$$\sigma_{\max}(B)^2\sigma_{\max}(C)^2 \geq \sigma_{\min}(B)^2\sigma_{\min}(C)^2 > 0.$$

Since for any $X_1, X_2 \in \mathbb{R}^{m \times n}$,

$$\|B^\top B(X_1 - X_2)CC^\top\|^2 \leq \sigma_{\max}(B)^2\sigma_{\max}(C)^2 \operatorname{tr}[(B^\top B(X_1 - X_2)CC^\top)^\top(X_1 - X_2)],$$

we have

$$(4.18) \quad \lambda \operatorname{tr}((X_{k+1} - X_*)^\top(Z_{k+1} - Z_*)) \geq \frac{1}{\sigma_{\max}(B)^2\sigma_{\max}(C)^2} \|\lambda Z_{k+1} - \lambda Z_*\|^2.$$

Taking a convex combination of (4.14) and (4.18), we get

$$(4.19) \quad \lambda \operatorname{tr}((X_{k+1} - X_*)^\top(Z_{k+1} - Z_*)) \geq t\sigma_{\min}(B)^2\sigma_{\min}(C)^2\|X_{k+1} - X_*\|^2 \\ + (1-t)\frac{1}{\sigma_{\max}(B)^2\sigma_{\max}(C)^2}\|\lambda Z_{k+1} - \lambda Z_*\|^2$$

for any $t \in [0, 1]$. When we derived (4.12), the term $2\sigma_{\min}(B)^2\sigma_{\min}(C)^2\|X_{k+1} - X_*\|^2/\lambda$ came from (4.14). If we use (4.19) in place of (4.14) in our derivation, we obtain

$$\|X_k - X_*\|^2 + \|Z_k - Z_*\|^2 - \|X_{k+1} - X_*\|^2 - \|Z_{k+1} - Z_*\|^2 \\ \geq \|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2 + \frac{2}{\lambda}\sigma_{\min}(B)^2\sigma_{\min}(C)^2\|X_k - X_{k+1}\|^2 \\ + t\frac{2}{\lambda}\sigma_{\min}(B)^2\sigma_{\min}(C)^2\|X_{k+1} - X_*\|^2 \\ + (1-t)\frac{2}{\lambda\sigma_{\max}(B)^2\sigma_{\max}(C)^2}\|\lambda Z_{k+1} - \lambda Z_*\|^2.$$

Now drop the nonnegative term $2\sigma_{\min}(B)^2\sigma_{\min}(C)^2\|X_k - X_{k+1}\|^2/\lambda$, and combine the last two terms to get $\delta(\|X_{k+1} - X_*\|^2 + \|Z_{k+1} - Z_*\|)$ with

$$t = \frac{\lambda^2}{1 + \sigma_{\min}(B)^2\sigma_{\min}(C)^2\sigma_{\max}(B)^2\sigma_{\max}(C)^2}. \quad \square$$

COROLLARY 4.5. *The iterate (X_k, Z_k) of Algorithm 5 satisfies*

$$\|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2 \leq \frac{\|X_0 - X_*\|^2 + \|Z_0 - Z_*\|^2}{k + 1}.$$

Proof. By subtracting two successive inequalities of the form (4.8), we get

$$\text{tr}((Z_{k+1} + X_{k+1} - X_k - Z_{k+2} - X_{k+2} + X_{k+1})^\top(Y_{k+2} - Y_{k+1})) \geq 0.$$

Adding this to $2\lambda \text{tr}((Z_{k+1} - Z_{k+2})^\top(X_{k+1} - X_{k+2})) \geq 0$ gives

$$\begin{aligned} &\text{tr}((X_{k+1} - X_{k+2})^\top(X_k - 2X_{k+1} + X_{k+2})) \\ &+ \text{tr}((Z_{k+1} + X_{k+1} - X_k - Z_{k+2} - X_{k+2} + X_{k+1})^\top(X_{k+1} - X_{k+2})) \geq 0. \end{aligned}$$

It follows from (4.4) that

$$Z_k + Y_{k+1} - X_k - Z_{k+1} - Y_{k+2} + X_{k+1} = Z_{k+1} + X_{k+1} - X_k - Z_{k+2} - X_{k+2} + X_{k+1}.$$

So the last inequality becomes

$$\begin{aligned} (4.20) \quad &\text{tr}((X_{k+1} - X_{k+2})^\top(X_k - 2X_{k+1} + X_{k+2})) \\ &+ \text{tr}((Z_k + Y_{k+1} - X_k - Z_{k+1} - Y_{k+2} + X_{k+1})^\top(X_{k+1} - Y_{k+1} - X_{k+2} + Y_{k+2})) \geq 0. \end{aligned}$$

Adding

$$\begin{aligned} &\|X_k - 2X_{k+1} + X_{k+2}\|^2 \\ &+ \text{tr}((X_k - Y_{k+1} - X_{k+1} + Y_{k+2})^\top(X_{k+1} - Y_{k+1} - X_{k+2} + Y_{k+2})) \end{aligned}$$

to both sides of (4.20), we get

$$\begin{aligned} &\text{tr}((X_k - X_{k+1})^\top(X_k - 2X_{k+1} + X_{k+2})) + \text{tr}((Z_k - Z_{k+1})^\top(Z_k - 2Z_{k+1} + Z_{k+2})) \\ &\geq \|X_k - 2X_{k+1} + X_{k+2}\|^2 \\ &+ \text{tr}((X_k - Y_{k+1} - X_{k+1} + Y_{k+2})^\top(X_{k+1} - Y_{k+1} - X_{k+2} + Y_{k+2})), \end{aligned}$$

and with this inequality, we see that

$$\begin{aligned} &\|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2 - \|X_{k+1} - X_{k+2}\|^2 - \|Z_{k+1} - Z_{k+2}\|^2 \\ &= 2 \text{tr}((X_k - X_{k+1})^\top(X_k - 2X_{k+1} + X_{k+2})) \\ &+ 2 \text{tr}((Z_k - Z_{k+1})^\top(Z_k - 2Z_{k+1} + Z_{k+2})) \\ &- \|X_k - 2X_{k+1} + X_{k+2}\|^2 - \|Z_k - 2Z_{k+1} + Z_{k+2}\|^2 \\ &\geq \|X_k - 2X_{k+1} + X_{k+2}\|^2 \\ &+ 2 \text{tr}((X_k - Y_{k+1} - X_{k+1} + Y_{k+2})^\top(X_{k+1} - Y_{k+1} - X_{k+2} + Y_{k+2})) \\ &- \|X_k - 2X_{k+1} + X_{k+2}\|^2 - \|Z_k - 2Z_{k+1} + Z_{k+2}\|^2 \\ &\geq 2\|X_k - 2X_{k+1} + X_{k+2}\|^2 \\ &+ 2 \text{tr}((X_k - 2X_{k+1} + X_{k+2} + Z_k - 2Z_{k+1} + Z_{k+2})^\top(Z_k - 2Z_{k+1} + Z_{k+2})) \\ &- \|X_k - 2X_{k+1} + X_{k+2}\|^2 - \|Z_k - 2Z_{k+1} + Z_{k+2}\|^2 \\ &= \|X_k - 2X_{k+1} + X_{k+2} + Z_k - 2Z_{k+1} + Z_{k+2}\|^2 \geq 0. \end{aligned}$$

It follows that the sequence $(\|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2)_{k=0}^\infty$ is monotone decreasing. Hence,

$$\begin{aligned} (k+1)(\|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2) &\leq \sum_{k=0}^{\infty} (\|X_k - X_{k+1}\|^2 + \|Z_k - Z_{k+1}\|^2) \\ &\leq \|X_0 - X_*\|^2 + \|Z_0 - Z_*\|^2, \end{aligned}$$

where the inequality follows from (4.12). \square

4.2. Repeated projections. There are occasions when we do not have a single closed-form solution for a projection onto the desired set $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ but we do have closed-form solutions for projections onto \mathcal{S}_1 and \mathcal{S}_2 . Of course, one may then obtain an iterative method for projection onto \mathcal{S} simply by alternating between projections onto \mathcal{S}_1 and \mathcal{S}_2 [8]. Nevertheless, in situations like this, standard wisdom from the design of iterative algorithms informs us that it would be better to intertwine these inexpensive projections onto \mathcal{S}_1 and \mathcal{S}_2 with other steps of Algorithm 5—instead of projecting onto \mathcal{S} , an expensive endeavor requiring a separate iterative procedure, in every iteration of Algorithm 5. With this in mind, we obtain the following “two projections variant” of Algorithm 5:

$$\begin{aligned} (4.21) \quad X_{k+1} &= \operatorname{argmin} \{ \|A - BXC\|^2 + \lambda \|X - W_k + Z_k\|^2 : X \in \mathbb{R}^{p \times q} \}, \\ Y_{k+1} &= \operatorname{argmin} \{ \|W_k - Z'_k - Y\| : Y \in \mathcal{D} \}, \\ W_{k+1} &= \operatorname{argmin} \{ \|X_{k+1} + Y_{k+1} + Z_k + Z'_k\|/2 - W\| : W \in \mathcal{S} \}, \\ Z_{k+1} &= Z_k + X_{k+1} - W_{k+1}, \\ Z'_{k+1} &= Z'_k + Y_{k+1} - W_{k+1}. \end{aligned}$$

It is straightforward to extend this to include three (or more) projections when we have $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$. In other words, we may solve a generalized matrix nearness problem $\min \|A - BXC\|$, where X is required to satisfy two or more of the constraints ①–⑥.

The convergence results in section 4.1 may also be adapted to (4.21). To account for the fact that we now have two Dykstra corrections Z_k and Z'_k , we replace (4.7) and (4.8) by

$$\begin{aligned} \lambda(W_k - W_{k+1} - Z_{k+1}) &= B^\top(BX_{k+1}C - A)C^\top, \\ \lambda \operatorname{tr}((W_k - W_{k+1} - Z'_{k+1})^\top(Y_{k+1} - Y)) &\geq 0, \\ \lambda \operatorname{tr}((Z_{k+1} + Z'_{k+1})^\top(W_{k+1} - W)) &\geq 0 \end{aligned}$$

for all $Y \in \mathcal{S}_1$ and $W \in \mathcal{S}_2$. It is straightforward to check that with this modification, the proofs of Lemma 4.2, Theorems 4.3 and 4.4, and Corollary 4.5 carry through for the algorithm in (4.21).

5. Computing projections. We rely on Algorithm 5 for the generalized nearness problems ⑤, ⑥, and the prescribed eigenvector problem. Since the algorithm alternates between projection and norm-constrained least squares, it remains to discuss how we may compute a projection, i.e.,

$$\min_{X \in \mathcal{S}} \|A - X\|,$$

for the relevant sets \mathcal{S} . We first remind the reader that the projection of $A \in \mathbb{R}^{n \times n}$ to the subspace of symmetric matrices $\mathbb{S}^n := \{X \in \mathbb{R}^{n \times n} : X = X^\top\}$ is given by $X = (A + A^\top)/2$, a fact that we will use liberally below.

For $\textcircled{5}$, \mathcal{S} is the subspace of either Toeplitz, Hankel, or circulant matrices [28]:

$$\begin{aligned} \text{Toep}_n(\mathbb{R}) &:= \{X \in \mathbb{R}^{n \times n} : x_{i,i+k} = x_{j,j+k}, -n+1 \leq k \leq n-1, \\ &\quad 1 \leq k+i, k+j, i, j \leq n\}, \\ \text{Hank}_n(\mathbb{R}) &:= \{X \in \mathbb{R}^{n \times n} : x_{i,k-i} = x_{j,k-j}, 2 \leq k \leq 2n, \\ &\quad 1 \leq k-i, k-j, i, j \leq n\}, \\ \text{Circ}_n(\mathbb{R}) &:= \{X \in \mathbb{R}^{n \times n} : x_{ij} = x_{kl}, i-j \equiv k-l \pmod n\}. \end{aligned}$$

The projections of $A \in \mathbb{R}^{n \times n}$ onto $\text{Toep}_n(\mathbb{R})$, $\text{Hank}_n(\mathbb{R})$, $\text{Circ}_n(\mathbb{R})$ are then given, respectively, by

$$x_{ij} = \begin{cases} \frac{1}{n-|i-j|} \sum_{k=l=i-j} a_{kl} & \text{(Toeplitz),} \\ \frac{1}{n-|i+j-n-1|} \sum_{k+l=i+j} a_{kl} & \text{(Hankel),} \\ \frac{1}{n} \sum_{k-l \equiv i-j \pmod n} a_{kl} & \text{(circulant)} \end{cases}$$

for $i, j = 1, \dots, n$.

For the prescribed eigenvector problem, given any nonzero $v \in \mathbb{R}^n$, we write

$$\mathbb{S}_v^n := \{X \in \mathbb{S}^n : Xv = \lambda v \text{ for some } \lambda \in \mathbb{R}\}.$$

For any given $A \in \mathbb{R}^{n \times n}$ and nonzero $v \in \mathbb{R}^n$, we may assume $\|v\|_2 = 1$ and, if not, just normalize. Let $V \in \mathbb{R}^{n \times n}$ be an orthogonal matrix whose first column is v . Let $e = [1, 0, \dots, 0]^T \in \mathbb{R}^n$. Then as $Xv = \lambda v$ iff $V^T X V e = \lambda e$, the projection problem reduces to

$$\min_{X \in \mathbb{S}_v^n} \|A - X\| = \min_{V^T X V \in \mathbb{S}_e^n} \|V^T A V - V^T X V\|.$$

With this observation, a projection of $A \in \mathbb{R}^{n \times n}$ to $X \in \mathbb{S}_v^n$ may be computed by first extending v to an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ (e.g., by using QR decomposition); partitioning

$$V^T A V = \begin{bmatrix} \alpha_{11} & a_{12}^T \\ a_{21} & A_{22} \end{bmatrix}$$

with $\alpha_{11} \in \mathbb{R}$, $a_{12}, a_{21} \in \mathbb{R}^{n-1}$, and $A_{22} \in \mathbb{R}^{(n-1) \times (n-1)}$; and finally computing

$$X = V \begin{bmatrix} \alpha_{11} & 0 \\ 0 & (A_{22} + A_{22}^T)/2 \end{bmatrix} V^T.$$

It remains to address the projections for $\textcircled{6}$. We introduce more standard notations: Nonnegative and positive semidefinite $X \in \mathbb{R}^{n \times n}$ are denoted

$$X \geq 0, \quad X \succeq 0,$$

respectively; i.e., the former means that $x_{ij} \geq 0$ for all $i, j = 1, \dots, n$, whereas the latter means that the quadratic form $v^T X v \geq 0$ for all $v \in \mathbb{R}^n$. The convex sets of nonnegative matrices and of symmetric positive semidefinite matrices are denoted

$$\mathbb{R}_+^{n \times n} := \{X \in \mathbb{R}^{n \times n} : X \geq 0\}, \quad \mathbb{S}_+^n := \{X \in \mathbb{S}^n : X \succeq 0\},$$

respectively. It is well known that the projections of a matrix $A \in \mathbb{R}^{n \times n}$ onto these sets are given by simply zeroing out negative entries or negative eigenvalues. For $\mathbb{R}_+^{n \times n}$, it is an easy exercise: The projected matrix $X \in \mathbb{R}_+^{n \times n}$ is given by

$$X = \max(A, 0),$$

where $\max(\cdot, 0)$ is applied coordinatewise to a matrix. For \mathbb{S}_+^n , it is slightly more involved [15]: The projected matrix $X \in \mathbb{S}_+^n$ is given by taking the symmetric eigenvalue decomposition $(A + A^T)/2 = V\Lambda V^T$ and setting

$$X = V \max(\Lambda, 0) V^T.$$

The same projected matrix X may also be computed with polar decomposition [15, 14].

The convex sets of stochastic, doubly stochastic, and correlation matrices [20] are

$$\begin{aligned} \text{Stoc}_n(\mathbb{R}) &:= \left\{ X \in \mathbb{R}_+^{n \times n} : \sum_{j=1}^n x_{ij} = 1, i = 1, \dots, n \right\}, \\ \text{DStoc}_n(\mathbb{R}) &:= \left\{ X \in \mathbb{R}_+^{n \times n} : \sum_{j=1}^n x_{ij} = \sum_{i=1}^n x_{ij} = 1, i, j = 1, \dots, n \right\}, \\ \text{Corr}_n(\mathbb{R}) &:= \{X \in \mathbb{S}_+^n : x_{ii} = 1, i = 1, \dots, n\}. \end{aligned}$$

By our discussion in section 4.2, we only need to address the question of projections onto

$$\begin{aligned} \mathcal{C} &:= \left\{ X \in \mathbb{R}^{n \times n} : \sum_{j=1}^n x_{ij} = 1, i = 1, \dots, n \right\}, \\ \mathcal{R} &:= \left\{ X \in \mathbb{R}^{n \times n} : \sum_{i=1}^n x_{ij} = 1, j = 1, \dots, n \right\}, \\ \mathcal{D} &:= \{X \in \mathbb{S}^n : x_{ii} = 1, i = 1, \dots, n\} \end{aligned}$$

since

$$\text{Stoc}_n(\mathbb{R}) = \mathcal{C} \cap \mathbb{R}_+^{n \times n}, \quad \text{DStoc}_n(\mathbb{R}) = \mathcal{C} \cap \mathcal{R} \cap \mathbb{R}_+^{n \times n}, \quad \text{Corr}_n(\mathbb{R}) = \mathcal{D} \cap \mathbb{S}_+^n.$$

The projections of $A \in \mathbb{R}^{n \times n}$ onto \mathcal{C} , \mathcal{R} , \mathcal{D} are easily seen to be given, respectively, by

$$x_{ij} = a_{ij} - \frac{1}{n} \left[\sum_{k=1}^n a_{ik} - 1 \right], \quad x_{ij} = a_{ij} - \frac{1}{n} \left[\sum_{k=1}^n a_{kj} - 1 \right], \quad x_{ij} = \begin{cases} 1, & i = j, \\ a_{ij} & i \neq j, \end{cases}$$

where $i, j = 1, \dots, n$.

6. Numerical experiments. We compare Algorithm 5, which is based on numerical linear algebra, with general algorithms based on convex optimization. We use problem © for illustration, as these nonnegativity constraints are more complex and allow us to test our two-projection variant (4.21). We will minimize $\|A - BXC\|$ with X constrained to nonnegative, stochastic, positive semidefinite, and correlation matrices. The first two require just a single projection, and thus Algorithm 5 suffices; the last two require two projections and thus call for (4.21). We compare our results against those obtained with convex optimization methods in the software package CVX

[5], using its ECOS solver for the nonnegative and stochastic cases and its SCS solver for the positive semidefinite and correlation cases.

We generate random matrices $B, C, X \in \mathbb{R}^{n \times n}$, with X satisfying the constraint at hand, and set $A = BXC$. This represents the most common scenario: We minimize $\|A - BXC\|$ when we really want to solve $A = BXC$ in the presence of errors. Furthermore, as X is known and (almost surely) unique, we may use the forward error $\|\hat{X} - X\|/\|X\|$ as a metric to make comparisons.

6.1. Speed. For each dimension $n = 2^d$, we repeat our runs 10 times and compare the average time taken to reach a prespecified forward error. The default precisions of CVX are $\text{epsrel} = \text{epsabs} = 1\text{e-}4$ for SCS and $\text{reltol} = \text{abstol} = \text{feastol} = 1\text{e-}8$ for ECOS; we scale these parameters by the dimension of the matrix n . We record the final forward error of CVX and run Algorithm 5 until it achieves the same forward error. From Figure 1, Algorithm 5 outperforms CVX significantly in speed for large n . Indeed, the range of dimensions is limited by CVX, which fails to converge for large n . To get a rough idea, for $n = 2^7$, CVX took about half an hour when Algorithm 5 took less than a second. For $n = 2^8$ and beyond, CVX did not converge within 24 hours.

These results are within expectation: In convex optimization, these problems are transformed to convex quadratic programs or semidefinite programs; both require at least $O(n^4)$ operations per iteration, which is prohibitive for large n . In our Algorithm 5 and its two-projection variant (4.21), the dominating cost is the projection onto \mathcal{S} —this is essentially free for nonnegative and stochastic matrices, requiring only

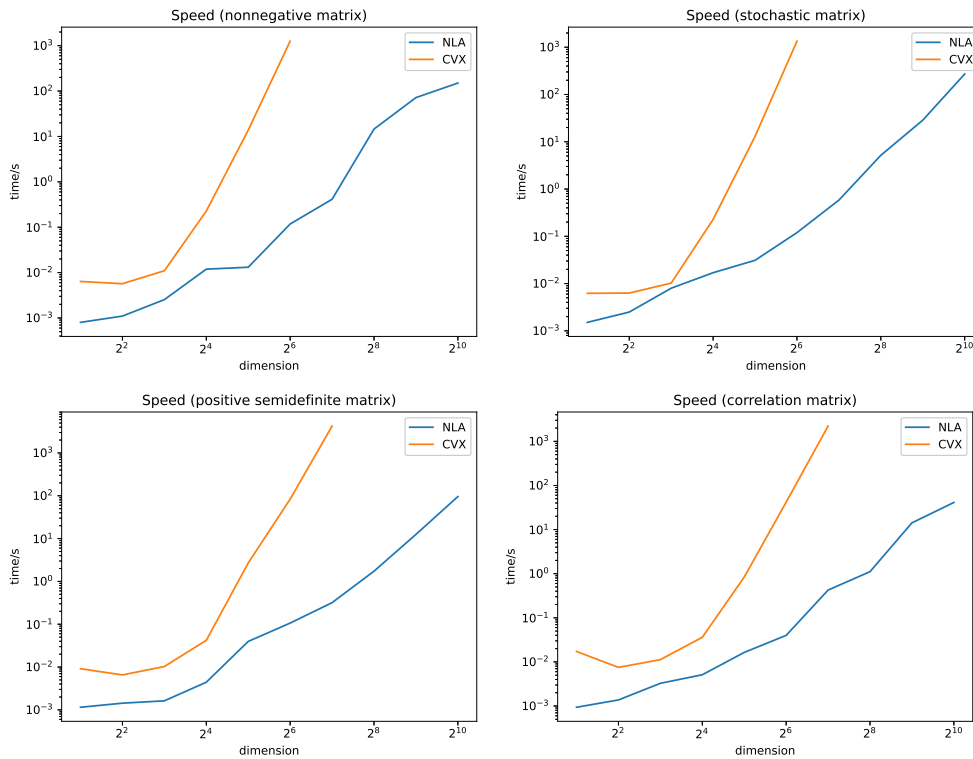


FIG. 1. Speed of Algorithm 5 (NLA) versus CVX.

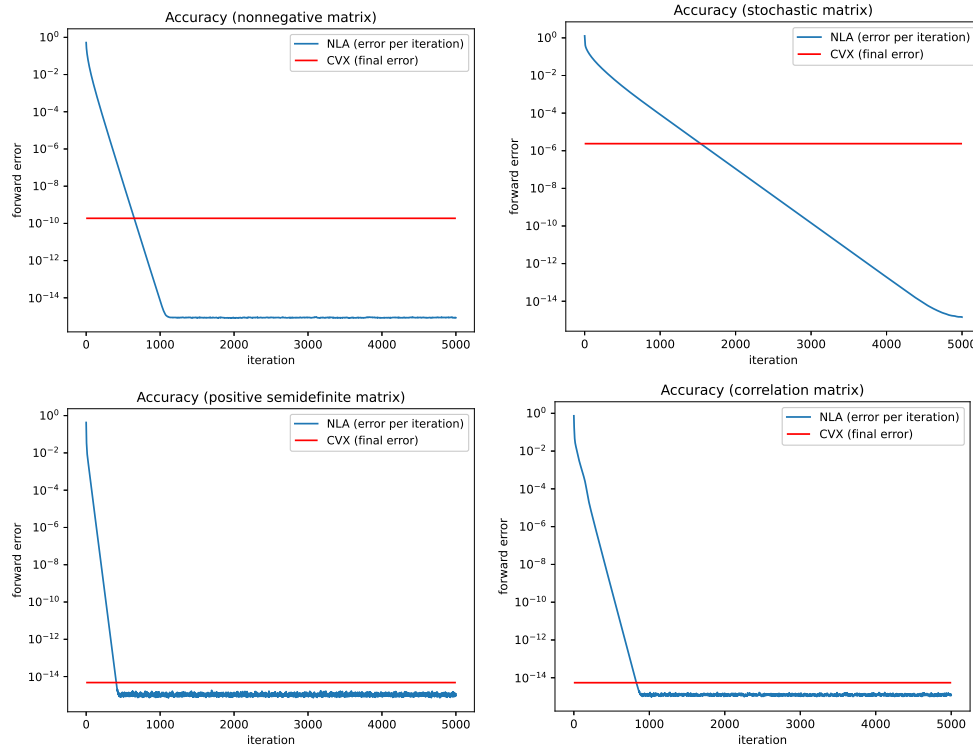


FIG. 2. Accuracy of Algorithm 5 (NLA) versus CVX. Here the red lines indicate the level of maximum possible accuracy of CVX.

$O(n^2)$ operations per iteration; for correlation and positive semidefinite matrices, projections require around $O(n^3)$ operations per iteration.

6.2. Accuracy. Here we compare the minimum possible forward error each method can achieve. We set $n = 32$, as CVX may fail to converge in a reasonable amount of time for larger values of n . We set CVX to its maximum allowed precisions—`reltol` = `abstol` = `feastol` = $1e-16$ for ECOS and `epsrel` = `epsabs` = $1e-16$ for SCS—and record its *final* forward error. Then we run Algorithm 5 for 5,000 iterations and record its forward error *at each iteration*. Note that it is not meaningful to make iterationwise comparisons here, as the two algorithms are entirely different. From Figure 2, we see that Algorithm 5 reaches beyond the maximum possible accuracy of CVX in every case. The linear convergence rate in Theorem 4.4 is also evident from these plots.

7. Conclusion. Likely because of the increasing awareness of convex optimization as a potent tool in many areas, there has been a tendency to apply general purpose convex optimization methods to any convex problem. However, convex problems like those considered in this article often have more structures than mere convexity. We show that approaching such problems through numerical linear algebra in the spirit of [11, 12, 15, 16, 17, 19, 22, 30, 35] can sometimes lead to better results and has the advantage of working for the occasional nonconvex problem like the ones with rank constraints or prescribed eigenvalues considered in this article.

A potential avenue for future work is to extend the work in this article to other unitarily invariant matrix norms: Closed-form expressions like those in sections 2 and 3, if possible, would likely require more advanced tools, such as singular value majorization [21, Chapter 3], used in the extension of the Eckart–Young theorem (for the Frobenius norm) to the Eckart–Young–Mirsky theorem (for any unitarily invariant norms). Iterative algorithms like the ones in section 4 would likely resemble proximal gradient algorithms [32] or eigenvalue optimization algorithms [25].

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REFERENCES

- [1] Å. BJÖRCK, *Numerical Methods for Least Squares Problems*, SIAM, Philadelphia, 1996, <https://doi.org/10.1137/1.9781611971484>.
- [2] J. E. BROCK, *Optimal matrices describing linear systems*, AIAA J., 6 (1968), pp. 1292–1296.
- [3] K.-W. E. CHU, *Singular value and generalized singular value decompositions and the solution of linear matrix equations*, Linear Algebra Appl., 88/89 (1987), pp. 83–98, [https://doi.org/10.1016/0024-3795\(87\)90104-2](https://doi.org/10.1016/0024-3795(87)90104-2).
- [4] K.-W. E. CHU, *Symmetric solutions of linear matrix equations by matrix decompositions*, Linear Algebra Appl., 119 (1989), pp. 35–50, [https://doi.org/10.1016/0024-3795\(89\)90067-0](https://doi.org/10.1016/0024-3795(89)90067-0).
- [5] CVX Research, Inc., *CVX: Matlab Software for Disciplined Convex Programming, Version 2.0*, 2012, <http://cvxr.com/cvx>.
- [6] J. W. DEMMEL, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, 1997, <https://doi.org/10.1137/1.9781611971446>.
- [7] F. J. H. DON, *On the symmetric solutions of a linear matrix equation*, Linear Algebra Appl., 93 (1987), pp. 1–7, [https://doi.org/10.1016/S0024-3795\(87\)90308-9](https://doi.org/10.1016/S0024-3795(87)90308-9).
- [8] R. L. DYKSTRA, *An algorithm for restricted least squares regression*, J. Amer. Statist. Assoc., 78 (1983), pp. 837–842.
- [9] C. ECKART AND G. YOUNG, *The approximation of one matrix by another of lower rank*, Psychometrika, 1 (1936), pp. 211–218.
- [10] S. FRIEDLAND AND A. TOROKHTI, *Generalized rank-constrained matrix approximations*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 656–659, <https://doi.org/10.1137/06065551>.
- [11] G. H. GOLUB, *Least squares, singular values and matrix approximations*, Appl. Math., 13 (1968), pp. 44–51.
- [12] G. H. GOLUB, *Some modified matrix eigenvalue problems*, SIAM Rev., 15 (1973), pp. 318–334, <https://doi.org/10.1137/1015032>.
- [13] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, 3rd ed., Johns Hopkins Studies in the Mathematical Sciences, The Johns Hopkins University Press, Baltimore, MD, 1996.
- [14] N. J. HIGHAM, *Computing the polar decomposition—With applications*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 1160–1174, <https://doi.org/10.1137/0907079>.
- [15] N. J. HIGHAM, *Computing a nearest symmetric positive semidefinite matrix*, Linear Algebra Appl., 103 (1988), pp. 103–118, [https://doi.org/10.1016/0024-3795\(88\)90223-6](https://doi.org/10.1016/0024-3795(88)90223-6).
- [16] N. J. HIGHAM, *The symmetric procrustes problem*, BIT, 28 (1988), pp. 133–143, <https://doi.org/10.1007/BF01934701>.
- [17] N. J. HIGHAM, *Matrix nearness problems and applications*, in Applications of Matrix Theory (Bradford, 1988), Institute of Mathematics and Its Applications Conference Series 22, Oxford University Press, Oxford, 1989, pp. 1–27, <https://doi.org/10.1093/imamat/22.1.1>.
- [18] N. J. HIGHAM, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002, <https://doi.org/10.1137/1.9780898718027>.
- [19] N. J. HIGHAM, *Computing the nearest correlation matrix—A problem from finance*, IMA J. Numer. Anal., 22 (2002), pp. 329–343, <https://doi.org/10.1093/imanum/22.3.329>.
- [20] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985, <https://doi.org/10.1017/CBO9780511810817>.
- [21] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1994.

- [22] J. B. KELLER, *Closest unitary, orthogonal and hermitian operators to a given operator*, Math. Mag., 48 (1975), pp. 192–197, <https://doi.org/10.2307/2690338>.
- [23] C. G. KHATRI AND S. K. MITRA, *Hermitian and nonnegative definite solutions of linear matrix equations*, SIAM J. Appl. Math., 31 (1976), pp. 579–585, <https://doi.org/10.1137/0131050>.
- [24] H. J. LARSON, *Least squares estimation of the components of a symmetric matrix*, Technometrics, 8 (1966), pp. 360–362, <https://doi.org/10.2307/1266370>.
- [25] A. S. LEWIS AND M. L. OVERTON, *Eigenvalue optimization*, in Acta Numerica, 1996, Acta Numerica 5, Cambridge University Press, Cambridge, 1996, pp. 149–190, <https://doi.org/10.1017/S0962492900002646>.
- [26] S. K. MITRA, *Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$* , Proc. Camb. Philos. Soc., 74 (1973), pp. 213–216, <https://doi.org/10.1017/s030500410004799x>.
- [27] C. C. PAIGE AND M. A. SAUNDERS, *Towards a generalized singular value decomposition*, SIAM J. Numer. Anal., 18 (1981), pp. 398–405, <https://doi.org/10.1137/0718026>.
- [28] V. Y. PAN, *Structured Matrices and Polynomials*, Springer-Verlag, Berlin, 2001, <https://doi.org/10.1007/978-1-4612-0129-8>.
- [29] M. H. PROTTER AND C. B. MORREY, JR., *A First Course in Real Analysis*, 2nd ed., Undergraduate Texts in Mathematics, Springer-Verlag, Berlin, 1991, <https://doi.org/10.1007/978-1-4419-8744-0>.
- [30] C. R. RAO, *Matrix approximations and reduction of dimensionality in multivariate statistical analysis*, in Multivariate Analysis 5, North-Holland, Amsterdam, 1980, pp. 3–22.
- [31] G. W. STEWART, *Matrix Algorithms*, Vol. 1, SIAM, Philadelphia, 1998, <https://doi.org/10.1137/1.9781611971408>.
- [32] K.-C. TOH AND S. YUN, *An accelerated proximal gradient algorithm for nuclear norm regularized linear least squares problems*, Pac. J. Optim., 6 (2010), pp. 615–640.
- [33] Q.-W. WANG, *A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity*, Linear Algebra Appl., 384 (2004), pp. 43–54, <https://doi.org/10.1016/j.laa.2003.12.039>.
- [34] J. H. WILKINSON, *Sensitivity of eigenvalues. II*, Utilitas Math., 30 (1986), pp. 243–286.
- [35] J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford, 1988.