SIMPLE MATRIX EXPRESSIONS FOR THE CURVATURES OF GRASSMANNIAN

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ABSTRACT. We show that modeling a Grassmannian as symmetric orthogonal matrices $\operatorname{Gr}(k, \mathbb{R}^n) \cong \{Q \in \mathbb{R}^{n \times n} : Q^\mathsf{T} Q = I, \ Q^\mathsf{T} = Q, \ \operatorname{tr}(Q) = 2k - n\}$ yields exceedingly simple matrix formulas for various curvatures and curvature-related quantities, both intrinsic and extrinsic. These include Riemann, Ricci, Jacobi, sectional, scalar, mean, principal, and Gaussian curvatures; Schouten, Weyl, Cotton, Bach, Plebański, cocurvature, nonmetricity, and torsion tensors; first, second, and third fundamental forms; Gauss and Weingarten maps; and upper and lower delta invariants. We will derive explicit, simple expressions for the aforementioned quantities in terms of standard matrix operations that are stably computable with numerical linear algebra. Many of these aforementioned quantities have never before been presented for the Grassmannian.

1. INTRODUCTION

While pure mathematicians typically abhor picking coordinates for manifolds, this is all but inevitable in applied mathematics. A good choice of extrinsic coordinates facilitates computations for the applied mathematician and, as we will see in this article, provides transparent, easy-to-calculate expressions that are useful even for investigations in pure mathematics.

For the Grassmannian of k-planes in $\mathbb{R}^n$, we showed in [27] that points on the manifold may be represented by matrices $Q \in \mathbb{R}^{n \times n}$ that are (i) symmetric $Q^\mathsf{T} = Q$, (ii) orthogonal $Q^\mathsf{T} Q = I$, (iii) involutive $Q^2 = I$. Clearly any two of these conditions imply the third and thus

$$(1) \qquad \operatorname{Gr}(k, \mathbb{R}^n) \cong \{Q \in \mathbb{S}^n : Q^2 = I, \ \operatorname{tr}(Q) = 2k - n\} =: \operatorname{Gr}(k, n)$$

where $\mathbb{S}^n$ denotes the Euclidean space of $n \times n$ symmetric matrices. Our motivation in [27] was largely computational — such a coordinate representation of points in $\operatorname{Gr}(k, \mathbb{R}^n)$ by orthogonal matrices gives immeasurably stabler numerical algorithms compared to other models of the Grassmannian as projection matrices or equivalence classes of matrices.

The goal of this article is to show that, even for calculations by hand, the *involution model* (1) provides a significant advantage over, say, expressions in [43], which are supposedly simple and already given in terms of linear algebra. Henceforth we define $\operatorname{Gr}(k, n)$ to be the set of matrices on the right of (1) to distinguish it from $\operatorname{Gr}(k, \mathbb{R}^n)$, the Grassmannian as an abstract manifold. In our earlier work [27], we derived expressions for basic quantities related to optimization: tangent vector, normal vector, metric, exponential map, geodesic, parallel transport, gradient, Hessian, etc, and showed that they all have simple, easily computable expressions in the involution model. Here we will do the same for various types of curvatures, some of which are notoriously difficult to calculate, but it is nevertheless rewarding as curvatures are likely the most important geometric objects of a Riemannian manifold. One might even argue that Riemannian geometry was created to provide a rigorous platform for studying curvatures.

A secondary goal is to illustrate the ease of using the involution model (1). We believe that many of the expressions derived in this article would be more difficult, some nearing impossible, to derive in other common models of the Grassmannian — as submanifolds of projective spaces, as various homogeneous spaces, or as a manifold of orthogonal projectors, all discussed in Section 7. Moreover, the expressions that we obtained are also more user-friendly, as we will elaborate below after presenting them in Table 1.

TABLE 1. The point $Q \in \mathrm{Gr}(k,n)$, tangent vectors $X, Y, Z, W \in \mathbb{T}_Q \mathrm{Gr}(k,n)$, and normal vector $H \in \mathbb{N}_Q \mathrm{Gr}(k,n)$ are parameterized as in (11).

CURVATURE	EXPRESSION	RESULT
first fundamental form	$\mathsf{I}(X,Y) = 2\operatorname{tr}(X_0^\mathsf{T} Y_0)$	Proposition 4.1
second fundamental form	$\mathsf{II}(X,Y) = \dfrac{1}{2}V\begin{bmatrix} X_0 Y_0^\mathsf{T} + Y_0 X_0^\mathsf{T} & 0 \\ 0 & -(X_0^\mathsf{T} Y_0 + Y_0^\mathsf{T} X_0) \end{bmatrix} V^\mathsf{T}$	Theorem 4.3
third fundamental form	$\mathsf{III}(X,Y) = -\dfrac{1}{2}\left(\dfrac{n}{2k(n-k)} + \dfrac{n-2}{4}\right)\operatorname{tr}(XY)$	Corollary 4.8
Gauss map	$\Gamma(Q) = \left\{ V\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} V^\mathsf{T} : H_1 \in \mathbb{S}^k,\ H_2 \in \mathbb{S}^{n-k} \right\}$	Proposition 4.2
Weingarten map	$\mathsf{S}(H)(X) = \dfrac{1}{2}V\begin{bmatrix} 0 & H_1 X_0 - X_0 H_2 \\ (H_1 X_0 - X_0 H_2)^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}$	Corollary 4.5
mean curvature vector	$\mathsf{H} = \dfrac{1}{2k(n-k)}V\begin{bmatrix} -(n-k)I_k & 0 \\ 0 & kI_{n-k} \end{bmatrix} V^\mathsf{T}$	Corollary 4.6
mean curvature	$\mathsf{H}(H) = \dfrac{(k-n)\operatorname{tr} H_1 + k\operatorname{tr} H_2}{2k(n-k)}$	Corollary 4.6
Gaussian curvature	$\mathsf{G}(H) = \dfrac{1}{2^{k(n-k)}}\prod_{i=1}^{k}\prod_{j=1}^{n-k}(\lambda_{k+j} - \lambda_i)$	Corollary 4.7
principal curvature	$\kappa_{ij}(H) = \dfrac{1}{2}(\lambda_{k+j} - \lambda_i),\quad i = 1,\ldots,k,\ j = 1,\ldots,n-k$	Corollary 4.7
Riemann curvature	$\mathsf{Rie}(X,Y,Z,W) = \dfrac{1}{2}\operatorname{tr}\big((XY - YX)ZW\big)$	Proposition 5.1
Jacobi curvature	$\mathsf{J}(X,Y,Z,W) = \operatorname{tr}(XYZW) - \operatorname{tr}\left(Y\left(\dfrac{XZ + ZX}{2}\right)W\right)$	Corollary 5.2
sectional curvature	$\kappa(X,Y) = \dfrac{1}{4}\dfrac{\|XY - YX\|^2}{\|X\|^2\|Y\|^2 - \operatorname{tr}(XY)^2}$	Corollary 5.3
Ricci curvature	$\mathsf{Ric}(X,Y) = \dfrac{n-2}{8}\operatorname{tr}(XY)$	Corollary 5.4
scalar curvature	$\mathsf{Sca} = \dfrac{k(n-k)(n-2)}{8}$	Corollary 5.4
traceless Ricci curvature	$\mathsf{Z}(X,Y) = 0$	Corollary 5.4
upper delta invariant	$\overline{\delta}_{2,r} = \dfrac{k(n-k)(n-2)}{8}$	Theorem 5.5
lower delta invariant	$\underline{\delta}_{2,r} = \dfrac{k(n-k)(n-2)}{8} - \dfrac{r}{4}$	Theorem 5.5
Schouten curvature	$\mathsf{P}(X,Y) = \dfrac{n-2}{16(k(n-k)-1)}\operatorname{tr}(XY)$	Corollary 5.6
Cotton curvature	$\mathsf{C}(X,Y,Z) = 0$	Corollary 5.7
Bach curvature	$\mathsf{B}(X,Y) = \dfrac{(n-2)^2}{32(k(n-k)-2)}\operatorname{tr}(XY)$	Corollary 5.9
Weyl curvature	$\mathsf{W}(X,Y,Z,W) = \dfrac{1}{2}\operatorname{tr}\big((XY - YX)ZW\big)$	Corollary 5.8
	$\qquad -\dfrac{(n-2)}{8(k(n-k)-1)}\big(\operatorname{tr}(XZ)\operatorname{tr}(YW) - \operatorname{tr}(XW)\operatorname{tr}(YZ)\big)$	

A few of the curvatures in Table 1 are extrinsic, i.e., they depend specifically on our model (1). These include the second and third fundamental forms; the Gauss and Weingarten maps; mean,

Gaussian, and principal curvatures. They help us better understand the embedded geometry of Grassmannian given by (1) but they also expedite our calculations of the intrinsic curvatures. These include the Riemann, Ricci, sectional, and scalar curvatures; the Schouten, Cotton, Weyl, and Bach tensors; and the upper and lower delta invariants.

While the value of an intrinsic curvature is independent of our choice of models, the expression or formula that gives this value is not. As is evident from Table 1, the involution model yields simple, stably computable formulas for extrinsic and intrinsic curvatures alike, essentially reducing curvatures of the Grassmannian to matrix analysis and their computations to numerical linear algebra. For example, computing the value of the Riemann curvature, a daunting order-4 tensor, is a trivial one-line calculation using our formula (and the proof of this formula is notably also a one-liner). For contrast, we will show in Section 7.2 what the corresponding calculation would entail if we use the most common Grassmannian model $\mathrm{O}(n)/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big)$.

What's new. For the intrinsic curvatures, the Schouten, Cotton, Weyl, Bach curvature tensors, the upper and lower delta invariants have never been explicitly calculated for a Grassmannian to the best of our knowledge. The formulas for Riemann, Ricci, sectional, and scalar curvatures of the Grassmannian modeled as various quotient spaces (see Section 7.2) are well-known and classical [9, 16, 40, 49] and they have also been calculated in [3, 32] for the projection model (see Section 7.3). The novelty of our calculations for these is that we derived intrinsic curvatures from extrinsic curvatures. This is why we will calculate extrinsic curvatures first.

The formulas for the extrinsic curvatures — second and third fundamental forms; Gauss and Weingarten maps; mean, Gaussian, and principal curvatures — are all new. Of course this is just a consequence of the relative obscurity of the involution model (1). Unlike intrinsic invariants, extrinsic ones are model-dependent, and it is expected that these have never been calculated for a new model. Also, extrinsic curvatures do not apply to the quotient models in Section 7.2 as they are only defined for embedded manifolds.

We emphasize that by 'formula' we mean an explicit expression like those in Table 1, involving actual matrices, and has no undetermined quantities. These curvatures may of course be expressed in terms of *local coordinates* or *equivalence classes* or *horizontal spaces*, but these invariably require additional computational overhead, which we will discuss in Section 7. Our formulas do not contain ambiguities that require further choices and effort to resolve.

In addition to the curvatures in Table 1, we will also discuss the cocurvature, nonmetricity, torsion, and Plebański tensors. We will see in Proposition 3.1 and Corollary 6.1 that they are trivially zero. We also proved in Corollary 4.4 that the index of relative nullity vanishes; in Corollary 6.2 that the third fundamental form, the Ricci curvature, the Schouten and Bach tensors are all Codazzi tensors; and in Corollary 6.4 that the Riemann and Weyl curvatures are divergence-free.

2. Notations and conventions

In this article, we use blackboard bold fonts for vector spaces (e.g., tangent and normal spaces, space of $n \times n$ symmetric matrices, etc.) and san serif fonts for all curvatures and curvature-related quantities (e.g., Table 1). The Riemann, Ricci, and scalar curvatures, arguably the three most important quantities, are given three-letter notations $\mathsf{Rie}, \mathsf{Ric}, \mathsf{Sca}$ for emphasis.

We write $\mathbb{E}^m$ for a Euclidean space of dimension m equipped with its Euclidean inner product $\langle \cdot, \cdot \rangle$. For concreteness, one may assume that this Euclidean space is $\mathbb{R}^n$ with $\langle x, y \rangle = x^\top y$, or $\mathbb{S}^n$ with $\langle X, Y \rangle = \mathrm{tr}(XY)$, or $\mathbb{R}^{m \times n}$ with $\langle X, Y \rangle = \mathrm{tr}(X^\top Y)$. We write $\mathrm{proj}_{\mathbb{W}} : \mathbb{E}^m \to \mathbb{E}^m$ for the orthogonal projection onto a subspace $\mathbb{W} \subseteq \mathbb{E}^m$. The space of all linear maps between vector spaces $\mathbb{V}$ and $\mathbb{W}$ will be denoted $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ with $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$ denoted specially as $\mathrm{End}(\mathbb{V})$. We write id for the identity map on any set.

For $X, Y \in \mathbb{R}^{n \times n}$, we write $[X, Y] = XY - YX$ for the commutator. We write $\mathfrak{so}(n) = \{X \in \mathbb{R}^{n \times n} : X^{\mathsf{T}} = -X\}$ for the special orthogonal Lie algebra, i.e., the set of skew-symmetric matrices with $[\,\cdot\,,\cdot\,]$ as its Lie bracket.

We write $\mathcal{M}$ for a smooth manifold, $C^\infty(\mathcal{M})$ for its ring of smooth real-valued functions, $\mathbb{T}_x\mathcal{M}$ and $\mathbb{N}_x\mathcal{M}$ for its tangent and normal spaces at $x \in \mathcal{M}$ respectively, and $\mathfrak{X}(\mathcal{M})$ for its $C^\infty(\mathcal{M})$-module of smooth vector fields. We denote vector fields with an arrow like $\vec{v}$. The word *tensor* in this article will always mean a tensor over a $C^\infty(\mathcal{M})$-module, i.e., a smooth tensor field on $\mathcal{M}$, and will be cast in the form of multilinear maps between tangent and normal spaces. With few exceptions, all multilinear maps in this article are defined at a specific point $x \in \mathcal{M}$. So as not to be overly verbose, we write "on/of $\mathcal{M}$" when we mean "on/of $\mathcal{M}$ at x" and we sometimes drop the subscript x like in Table 1 when there is no cause for confusion.

We write $\mathrm{Gr}(k, \mathbb{V})$ for the Grassmannian of k-dimensional subspaces in the vector space $\mathbb{V}$. We emphasize that in this article,

$$(2) \qquad\qquad \mathrm{Gr}(k, n) \coloneqq \{Q \in \mathbb{S}^n : Q^2 = I, \ \mathrm{tr}(Q) = 2k - n\},$$

i.e., $\mathrm{Gr}(k, n) \subseteq \mathbb{S}^n$ is the image of the embedding

$$(3) \qquad\qquad \varepsilon : \mathrm{Gr}(k, \mathbb{R}^n) \to \mathbb{S}^n, \quad \mathbb{W} \mapsto P_{\mathbb{W}} - P_{\mathbb{W}^\perp} =: Q_{\mathbb{W}},$$

where $P_{\mathbb{W}} \in \mathbb{S}^n$ is the orthogonal projection matrix with image $\mathbb{W}$. So ε sends a k-dimensional subspace $\mathbb{W} \subseteq \mathbb{R}^n$ to a matrix $Q_{\mathbb{W}} \in \mathbb{S}^n$. It is easy to verify [27] that $Q_{\mathbb{W}}$ has the properties in (2), ε gives an embedding of Riemannian manifolds, and

$$\mathrm{Gr}(k, n) = \varepsilon\big(\mathrm{Gr}(k, \mathbb{R}^n)\big).$$

3. Curvature zoo

We will review the definitions of various curvatures and curvature-related quantities. This section is not intended to be pedagogical, and only contains minimal commentaries. The goal is just to collect definitions scattered across standard references [5, 16, 24, 25, 26, 30, 38] and some slightly less standard ones [10, 11, 15, 22, 37] and present them in a unified set of notations (see Section 2) for the reader's easy reference.

All discussions below assume that $\mathcal{M}$ is a Riemannian manifold with Riemannian metric $\mathbf{g}$, i.e., $\mathbf{g}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}$ defines an inner product at $x \in \mathcal{M}$. Section 3.2 applies to $\mathcal{M}$ intrinsically. Section 3.1 applies when $\mathcal{M}$ embedded in an Euclidean space $\mathbb{E}^m$. More precisely by embedding we always mean an isometric embedding $\varepsilon : \mathcal{M} \to \mathbb{E}^m$ preserving Riemannian metric, i.e., $\mathbf{g}_x(v, w) = \langle d_x\varepsilon(v), d_x\varepsilon(w)\rangle$ for all $x \in \mathcal{M}$, $v, w \in \mathbb{T}_x\mathcal{M}$. Henceforth, we will identify $\mathcal{M}$ with its image under the embedding ε so that we have $\mathcal{M} \subseteq \mathbb{E}^m$ and $\mathbf{g}_x(v, w) = \langle v, w\rangle$.

As usual, we will use the Levi-Civita connection throughout. We will say a few words about this choice in Section 3.3

3.1. Extrinsic curvatures.
In this section, $\mathcal{M} \subseteq \mathbb{E}^m$ is an n-dimensional submanifold of an m-dimensional Euclidean space. For any $x \in \mathcal{M}$, we have the canonical identification

$$\mathbb{E}^m \cong \mathbb{T}_x\mathbb{E}^m = \mathbb{T}_x\mathcal{M} \oplus \mathbb{N}_x\mathcal{M}$$

and the corresponding orthogonal projections $\mathrm{proj}_{\mathbb{T}_x\mathcal{M}}$ and $\mathrm{proj}_{\mathbb{N}_x\mathcal{M}}$.

The *first fundamental form* is

$$\mathrm{I}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathrm{I}_x(v, w) \coloneqq \langle v, w\rangle.$$

This is nothing more than the Riemannian metric $\mathbf{g}$ on $\mathcal{M}$ expressed in terms of the inner product $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathbb{E}^m$.

The *Gauss map* is defined by

$$\Gamma : \mathcal{M} \to \mathrm{Gr}(m - n, \mathbb{E}^m), \quad \Gamma(x) \coloneqq \mathbb{N}_x\mathcal{M}.$$

Take any $\mathbb{V} \in \operatorname{Gr}(m-n, \mathbb{E}^m)$, note that this is an $(m-n)$-dimensional subspace of $\mathbb{E}^m$ and we have

$$\mathbb{T}_{\mathbb{V}} \operatorname{Gr}(m-n, \mathbb{E}^m) \cong \operatorname{Hom}(\mathbb{V}, \mathbb{V}^{\perp}).$$

In particular, $\mathbb{T}_{\mathbb{N}_x\mathcal{M}} \operatorname{Gr}(m-n, \mathbb{E}^m) \cong \operatorname{Hom}(\mathbb{T}_x\mathcal{M}, \mathbb{N}_x\mathcal{M})$ and we may also regard the Gauss map as

(4) $$\Gamma : \mathcal{M} \to \operatorname{Hom}(\mathbb{T}_x\mathcal{M}, \mathbb{N}_x\mathcal{M}).$$

The *second fundamental form* II_x of $\mathcal{M}$ is given by the derivative of the Gauss map in the form of (4), i.e.,

$$\mathrm{II}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{N}_x\mathcal{M}, \quad \mathrm{II}_x(v, w) := d_x\Gamma(v)(w).$$

This is likely the most important extrinsic differential geometric invariant of an embedded manifold. Indeed many of our curvatures, including intrinsic ones, will be derived from the second fundamental form. We define the *index of relative nullity* [15] of $\mathcal{M}$ as

$$\nu_x := \dim\{v \in \mathbb{T}_x\mathcal{M} : \mathrm{II}_x(v, w) = 0 \text{ for all } w \in \mathbb{T}_x\mathcal{M}\},$$

noting that $\mathrm{II}_x(v, w) = \mathrm{II}_x(w, v)$. For each $\eta \in \mathbb{N}_x\mathcal{M}$, we may regard $\langle \mathrm{II}_x, \eta \rangle$ as an endomorphism on $\mathbb{T}_x\mathcal{M}$ and this is called the *Weingarten map* or *shape operator*,

$$\mathsf{S}_x(\eta) : \mathbb{T}_x\mathcal{M} \to \mathbb{T}_x\mathcal{M}, \quad \langle \mathsf{S}_x(\eta)(v), w \rangle := \langle \mathrm{II}_x(v, w), \eta \rangle.$$

This operator is self-adjoint as the second fundamental form is symmetric. The eigenvalues (necessarily real)

$$\lambda_1(\eta), \ldots, \lambda_n(\eta) \in \mathbb{R},$$

are called the *principal curvatures* of $\mathcal{M}$ along η. Their product is called the *Gaussian curvature* of $\mathcal{M}$ along η

$$\mathsf{G}_x(\eta) := \det \mathsf{S}_x(\eta) \in \mathbb{R},$$

and their sum is called the *mean curvature* of $\mathcal{M}$ along η

$$\mathsf{H}_x(\eta) := \operatorname{tr} \mathsf{S}_x(\eta) \in \mathbb{R}.$$

For any orthonormal basis $\eta_1, \ldots, \eta_{m-n} \in \mathbb{N}_x\mathcal{M}$, the *mean curvature vector* of $\mathcal{M}$ is

$$\mathsf{H}_x := \sum_{i=1}^{m-n} \mathsf{H}_x(\eta_i)\eta_i \in \mathbb{N}_x\mathcal{M}$$

and its value is independent of the choice of orthonormal basis. Clearly, $\mathsf{H}_x(\eta) = \langle \mathsf{H}_x, \eta \rangle$.

The *Gauss–Obata* map [37] of $\mathcal{M}$ is

$$\mathsf{Q}_x : \mathbb{T}_x\mathcal{M} \to \mathbb{T}_x\mathcal{M}, \quad \mathsf{Q}_x(v) := \sum_{j=1}^{m-n} \mathsf{S}_x(\eta_j)^2(v).$$

and the *third fundamental form* is

$$\mathrm{III}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathrm{III}_x(v, w) := \langle \mathsf{Q}_x(v), w \rangle.$$

The version defined here differs from an alternative version defined in classical differential [14, Equation 21] and algebraic [20, Equations 1.45 and 1.46] geometry. The latter allows for kth fundamental forms for all $k \geq 4$. Nevertheless the important thing is that both versions agree with the classical third fundamental form for a surface $\mathcal{M} \subseteq \mathbb{R}^3$.

Getting slightly ahead of ourselves, the second fundamental form may also be expressed using the Levi-Civita connection ∇ in (6):

(5) $$\mathrm{II}_x(v, w) = \partial_v \vec{w} - \nabla_v \vec{w} = \operatorname{proj}_{\mathbb{N}_x\mathcal{M}}(\partial_v \vec{w})$$

for any $v, w \in \mathbb{T}_x\mathcal{M}$ and where $\vec{w} \in \mathfrak{X}(\mathcal{M})$ is any vector field with $\vec{w}(x) = w$ [30, Theorem 8.2]. The value of $\mathrm{II}_x(v, w)$ is independent of the choice of $\vec{w}$ [30, Proposition 8.1].

3.2. **Intrinsic curvatures.** In this section $\mathcal{M}$ is a Riemannian manifold with metric tensor g. One feature of our approach is that we will calculate intrinsic curvatures in extrinsic coordinates given by our involution model, vastly simplifying the work involved. As such it suffices to define the *Levi-Civita connection* ∇ of $\mathcal{M}$ as an embedded manifold:

$$(6) \qquad \nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}), \quad (\nabla(\vec{v}, \vec{w}))(x) := \mathrm{proj}_{\mathbb{T}_x\mathcal{M}}(\partial_v \vec{w}),$$

where $v := \vec{v}(x) \in \mathbb{T}_x\mathcal{M}$ and $\partial_v \vec{w}$ is the standard directional derivative of the vector field, i.e., derivative of the vector-valued function $\vec{w} : \mathcal{M} \to \mathbb{E}^m$ along the direction $v \in \mathbb{T}_x\mathcal{M} \subseteq \mathbb{E}^m$. Note that this simple definition is possible only because both $\mathcal{M}$ and $\mathbb{T}_x\mathcal{M}$ are regarded as subsets of $\mathbb{E}^m$. It is also common to write, for a fixed $v \in \mathbb{T}_x\mathcal{M}$,

$$\nabla_v \vec{w} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}), \quad (\nabla_v \vec{w})(x) := (\nabla(\vec{v}, \vec{w}))(x),$$

as it behaves like a directional derivative. A slight variation of this notation makes $v \in \mathbb{T}_x\mathcal{M}$ the variable and fixes $x \in \mathcal{M}$, giving

$$_x\nabla \vec{w} : \mathbb{T}_x\mathcal{M} \to \mathbb{T}_x\mathcal{M}, \quad (_x\nabla \vec{w})(v) := (\nabla_v \vec{w})(x).$$

Since $_x\nabla \vec{w}$ is a linear operator, it has a trace, which defines the *divergence* for a vector field $\vec{w}$,

$$\mathrm{div} : \mathfrak{X}(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad \mathrm{div}(\vec{w})(x) := \mathrm{tr}(_x\nabla \vec{w}).$$

For higher-order tensor fields, the convention is to apply divergence to the last argument: If $\vec{w}_1, \ldots, \vec{w}_k \in \mathfrak{X}(\mathcal{M})$, then

$$\mathrm{div}(\vec{w}_1 \otimes \cdots \otimes \vec{w}_{k-1} \otimes \vec{w}_k) := \mathrm{div}(\vec{w}_k)\vec{w}_1 \otimes \cdots \otimes \vec{w}_{k-1},$$

and extended linearly to all k-tensor fields [38].

We will need two common notions [5] defined for any symmetric bilinear forms on $\mathbb{T}_x\mathcal{M}$. Let $\alpha, \beta : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}$ be symmetric and bilinear, their *Kulkarni–Nomizu product* $\alpha \owedge \beta$ is the symmetric quadrilinear form

$$(7) \qquad \begin{array}{c} \alpha \owedge \beta : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \\ \alpha \owedge \beta(u, v, w, z) := \alpha(u, w)\beta(v, z) - \alpha(u, z)\beta(v, w) - \alpha(v, w)\beta(u, z) + \alpha(v, z)\beta(u, w). \end{array}$$

If a symmetric bilinear form $\beta : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}$ satisfies

$$(\nabla_u \beta)(v, w) = (\nabla_v \beta)(u, w)$$

for all $u, v, w \in \mathbb{T}_x\mathcal{M}$, then it is called a *Codazzi tensor*.

The *Riemann curvature* or *curvature tensor* of $\mathcal{M}$ is

$$\mathsf{Rie}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{Rie}_x(u, v, w, z) := \langle \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]}w, z \rangle.$$

There is a common variant known by the same name:

$$\mathsf{R}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathrm{End}(\mathbb{T}_x\mathcal{M}), \quad \mathsf{R}_x(u, v)w := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]}w.$$

Note that $\mathsf{Rie}_x(u, v, w, z) = \langle \mathsf{R}_x(u, v)w, z \rangle$ with the slight difference being that R_x is a bilinear map and Rie_x is a quadrilinear form. There is also a symmetric variant called the *Jacobi tensor* of $\mathcal{M}$,

$$\mathsf{J}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{J}_x(u, v, w, z) := \frac{1}{2}\left(\mathsf{Rie}_x(u, v, w, z) + \mathsf{Rie}_x(w, v, u, z)\right).$$

If $v, w \in \mathbb{T}_x\mathcal{M}$ are linearly independent, then the *sectional curvature* of $\mathcal{M}$ is

$$\kappa_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \kappa_x(v, w) := \frac{\mathsf{Rie}_x(v, w, w, v)}{\|v \wedge w\|^2} = \frac{\mathsf{Rie}_x(v, w, w, v)}{\|v\|^2 \|w\|^2 - \langle v, w \rangle^2}.$$

Let $v_1, \ldots, v_n \in \mathbb{T}_x\mathcal{M}$ be an orthonormal basis. The *Ricci curvature* of $\mathcal{M}$ is

$$\mathsf{Ric}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{Ric}_x(v, w) := \sum_{j=1}^n \mathsf{Rie}_x(v, v_j, v_j, w).$$

The *scalar curvature* of $\mathcal{M}$ is

$$\mathsf{Sca}_x := \mathrm{tr}(\mathsf{Ric}_x) = \sum_{1 \le j < k \le n} \kappa_x(v_j, v_k) = \sum_{1 \le j < k \le n} \mathsf{Rie}_x(v_j, v_k, v_k, v_j).$$

The last two curvatures gives us the *traceless Ricci curvature*,

$$\mathsf{Z}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{Z}_x(v, w) := \mathsf{Ric}_x(v, w) - \frac{\mathsf{Sca}_x}{n}\mathsf{g}_x(v, w),$$

important as it gives a g-orthogonal decomposition of Ricci curvature. A manifold with $\mathsf{Z} = 0$ is called an Einstein manifold [5].

The scalar curvature allows a generalization to any d-dimensional subspace $\mathbb{V} \subseteq \mathbb{T}_x\mathcal{M}$,

$$\mathsf{Sca}_x(\mathbb{V}) := \sum_{1 \le j < k \le d} \kappa_x(v_j, v_k),$$

where $v_1, \dots, v_d \in \mathbb{V}$ is any orthonormal basis. Clearly $\mathsf{Sca}_x(\mathbb{T}_x\mathcal{M}) = \mathsf{Sca}_x$. From these one may construct the upper and lower *delta invariants* [10, 11], given respectively by

$$\overline{\delta}_x(d_1, \dots, d_r) := \mathsf{Sca}_x - \inf_{\substack{\dim \mathbb{V}_j = d_j \\ \mathbb{V}_j \perp \mathbb{V}_k, \ j < k}} \left[\sum_{j=1}^r \mathsf{Sca}_x(\mathbb{V}_j) \right],$$

$$\underline{\delta}_x(d_1, \dots, d_r) := \mathsf{Sca}_x - \sup_{\substack{\dim \mathbb{V}_j = d_j \\ \mathbb{V}_j \perp \mathbb{V}_k, \ j < k}} \left[\sum_{j=1}^r \mathsf{Sca}_x(\mathbb{V}_j) \right],$$

where $d_1, \dots, d_r \in \mathbb{Z}$ are such that $2 \le d_1 \le \cdots \le d_r$ and $d_1 + \cdots + d_r \le \dim \mathcal{M}$.

We will next define a well-known quartet of closely related curvature tensors [5]. The *Schouten tensor* of $\mathcal{M}$ is

$$\mathsf{P}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{P}_x(v, w) := \frac{1}{n-2}\left(\mathsf{Ric}_x(v, w) - \frac{\mathsf{Sca}_x}{2(n-1)}\mathsf{g}_x(v, w) \right).$$

The *Cotton tensor* of $\mathcal{M}$ is

$$\mathsf{C}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{C}_x(u, v, w) := (\nabla_u \mathsf{P}_x)(v, w) - (\nabla_v \mathsf{P}_x)(u, w).$$

The *Weyl tensor* of $\mathcal{M}$ is

$$\mathsf{W}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{W}_x := \mathsf{Rie}_x - \frac{1}{n-2}\mathsf{Z}_x \oslash \mathsf{g}_x - \frac{\mathsf{Sca}_x}{2n(n-1)}\mathsf{g}_x \oslash \mathsf{g}_x.$$

The *Bach* tensor of $\mathcal{M}$ is

$$\mathsf{B}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R},$$

$$\mathsf{B}_x(u, w) := \frac{1}{n-3} \sum_{i,j=1}^n \nabla^2_{v_i, v_j} \mathsf{W}_x(u, v_i, v_j, w) + \frac{1}{n-2} \sum_{i,j=1}^n \mathsf{Ric}_x(v_i, v_j)\mathsf{W}_x(u, v_i, v_j, w),$$

We will also describe some intrinsic curvatures that are more typically studied in non-Riemannian geometry, i.e., for a connection ∇ other than the Levi-Civita connection (see Section 3.3).

The *torsion tensor* [24, Chapter III, Section 5] of $\mathcal{M}$ is

$$\mathsf{T}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{T}_x(v, w) := \nabla_v w - \nabla_w v - [v, w].$$

The *nonmetricity tensor* [22] of $\mathcal{M}$ is

$$\mathsf{Q}_x : \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \times \mathbb{T}_x\mathcal{M} \to \mathbb{R}, \quad \mathsf{Q}_x(u, v, w) := -\nabla_u \mathsf{g}_x(v, w) + \mathsf{g}_x(\nabla_u v, w) + \mathsf{g}_x(v, \nabla_u w).$$

For any vector subbundle $\mathbb{V}\mathcal{M}$ of $\mathbb{T}\mathcal{M}$ with projection $\pi : \mathbb{T}\mathcal{M} \to \mathbb{V}\mathcal{M}$, its *cocurvature* [35, 26] is

$$\mathsf{R}^*_\pi : \mathbb{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}), \quad \mathsf{R}^*_\pi(\vec{v}, \vec{w}) := (\mathrm{id} - \pi)\big([\pi(\vec{v}), \pi(\vec{w})]\big).$$

In general R^*_π measures the failure of integrability of $\mathbb{V}\mathcal{M}$. To put the cocurvature in perspective, the *curvature* in this context is

$$R_\pi : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}), \quad R_\pi(\vec{v}, \vec{w}) := \pi\big([(\mathrm{id} - \pi)(\vec{v}), (\mathrm{id} - \pi)(\vec{w})]\big).$$

Evidently, this is a more general notion than the Riemann curvature R but we will see how they are related in Proposition 3.1.

3.3. Connections.
Our choice of Levi-Civita connection is all but preordained by the Fundamental Theorem of Riemannian Geometry [16, Chapter 2, Theorem 3.6], namely, it is the unique affine connection that is torsion-free and metric-compatible. To reassure the readers, we will add a few more words to justify this choice.

In manifold optimization, one needs a metric to identify tangent space with cotangent space, and the Levi-Civita connection is the most natural one (i.e., torsion free) compatible with our Riemannian metric g. While the goal of our article is not optimization per se, there are not many options among other common connections either: (i) The Weyl connection is a generalization of Levi-Civita connection to conformal metrics and for our choice of g the two are identical. (ii) The affine, Cartan, Ehresmann, and Koszul connections are unnatural in our context. (iii) The Gauss–Manin and Grothendieck connections are intended for schemes and incompatible with our consideration of the Grassmannian as a manifold. (iv) Other connections like those of Connes and Weitzenböck are even further removed from our treatment in this article.

Unsurprisingly the non-Riemannian curvatures all turn out to be trivial for a Riemannian manifold. The statement (a) below is just stated for ease of referencing.

Proposition 3.1. *Let ∇ be the Levi-Civita connection on $\mathcal{M}$.*

(a) *The torsion tensor and the nonmetricity tensor vanishes identically, i.e.,*

$$\mathsf{T}_x(v, w) = 0, \qquad \mathsf{Q}_x(u, v, w) = 0$$

for all $x \in \mathcal{M}$ and all $u, v, w \in \mathbb{T}_x\mathcal{M}$.

(b) *Let $\pi : \mathrm{O}(\mathcal{M}) \to \mathcal{M}$ be the orthonormal frame bundle on $\mathcal{M}$, $\mathbb{V}\mathcal{M} := \ker(d\pi)$, and $\widehat{\pi} : \mathbb{T}\,\mathrm{O}(\mathcal{M}) \to \mathbb{V}\mathcal{M}$ the projection induced by ∇. Then the cocurvature vanishes and the curvature equals the Riemann curvature up to sign, i.e.,*

$$\mathsf{R}^*_{\widehat{\pi}}(\vec{v}, \vec{w}) = 0, \qquad \mathsf{R}_{\widehat{\pi}}(\vec{v}, \vec{w}) = -\mathsf{R}(\vec{v}, \vec{w})$$

where the first equality holds for all $\vec{v}, \vec{w} \in \mathfrak{X}(\mathrm{O}(\mathcal{M}))$, the second for all $\vec{v}, \vec{w} \in \mathfrak{X}(\mathcal{M}) \subseteq \mathfrak{X}(\mathrm{O}(\mathcal{M}))$.

Proof. Since the Levi-Civita connection on $\mathcal{M}$ is, by definition, the unique connection that is torsion free and compatible with the metric g, both T and Q vanish.

The horizontal bundle $\ker \widehat{\pi}$ is $\mathbb{T}\mathcal{M}$, whose integral manifold is $\mathcal{M}$. So the cocurvature $\mathsf{R}^*_{\widehat{\pi}}$ must vanish identically. Since $\mathrm{O}(\mathcal{M})$ consists of orthonormal frames on $\mathcal{M}$, the typical fiber of $\mathbb{V}(\mathcal{M}) = \ker(d\pi)$ is $\mathfrak{so}(n)$, where $n = \dim \mathcal{M}$. As $\widehat{\pi}$ is induced by ∇, it can be regarded as an $\mathfrak{so}(n)$-valued differential 1-form ω on $\mathrm{O}(\mathcal{M})$. The projection map $\widehat{\pi}$ gives a decomposition

$$\mathbb{T}\,\mathrm{O}(\mathcal{M}) \simeq \mathbb{V}\mathcal{M} \oplus \mathbb{T}\mathcal{M}.$$

Thus for any $\vec{v}, \vec{w} \in \mathfrak{X}(\mathrm{O}(\mathcal{M}))$, we may identify $(\mathrm{id} - \widehat{\pi})(\vec{v})$ and $(\mathrm{id} - \widehat{\pi})(\vec{w})$ with elements in $\mathfrak{X}(\mathcal{M})$ and

$$-\mathsf{R}_{\widehat{\pi}}(\vec{v}, \vec{w}) = -\omega([(\mathrm{id} - \widehat{\pi})(\vec{v}), (\mathrm{id} - \widehat{\pi})(\vec{w})]) = d\omega\big((\mathrm{id} - \widehat{\pi})(\vec{v}), (\mathrm{id} - \widehat{\pi})(\vec{w})\big).$$

In other words, $-\mathsf{R}_{\widehat{\pi}}$ turns out to be the curvature 2-form of ω, which by [24, Section III.5] is equal to the Riemann curvature R. □

4. Extrinsic curvatures of the Grassmannian

We are now in a position to calculate various curvatures of the Grassmannian modeled as $\mathrm{Gr}(k,n)$ and express them as simple matrix formulas. Our strategy is first calculate the extrinsic curvatures in this section, notably the second fundamental form, and then use it as the basis for our calculation of intrinsic curvatures in Section 5.

Our ambient Euclidean space of choice is $\mathbb{S}^n$ equipped with the standard (also called trace or Frobenius) inner product on $\mathbb{R}^{m \times n}$ given by

$$\langle X, Y \rangle := \mathrm{tr}(X^\mathsf{T}Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}y_{ij}.$$

When restricted to $\mathrm{Gr}(k,n)$, it gives us our Riemannian metric

$$(8) \qquad \mathsf{g}_Q : \mathbb{T}_Q\,\mathrm{Gr}(k,n) \times \mathbb{T}_Q\,\mathrm{Gr}(k,n) \to \mathbb{R}, \quad \mathsf{g}_Q(X,Y) = \mathrm{tr}(X^\mathsf{T}Y)$$

for all $Q \in \mathrm{Gr}(k,n)$. Of course, we have $X^\mathsf{T} = X$ as $X \in \mathbb{S}^n$ but we choose to keep the transpose in our notation to remind ourselves that this is the trace inner product.

Given $Q \in \mathrm{Gr}(k,n) \subseteq \mathbb{S}^n$, we have an eigenvalue decomposition $Q = VI_{k,n-k}V^\mathsf{T}$ for some $V \in \mathrm{O}(n)$ and $I_{k,n-k} := \mathrm{diag}(I_k, -I_{n-k}) = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$. The tangent and normal spaces of $\mathrm{Gr}(k,n)$ at Q are

$$(9) \qquad \mathbb{T}_Q\,\mathrm{Gr}(k,n) = \left\{ V \begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T} \in \mathbb{S}^n : X_0 \in \mathbb{R}^{k \times (n-k)} \right\},$$

$$(10) \qquad \mathbb{N}_Q\,\mathrm{Gr}(k,n) = \left\{ V \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} V^\mathsf{T} \in \mathbb{S}^n : H_1 \in \mathbb{S}^k,\ H_2 \in \mathbb{S}^{n-k} \right\}.$$

Henceforth we will consistently write any point $Q \in \mathrm{Gr}(k,n)$, tangent vector $X \in \mathbb{T}_Q\,\mathrm{Gr}(k,n)$ and normal vector $H \in \mathbb{N}_Q\,\mathrm{Gr}(k,n)$ as

$$(11) \qquad Q = V \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} V^\mathsf{T}, \quad X = V \begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}, \quad H = V \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} V^\mathsf{T}.$$

The simple parameterization of these three basic objects in the involution model is a key to the simplicity of our calculations. A convenient orthonormal basis of $\mathbb{T}_{I_{k,n-k}}\,\mathrm{Gr}(k,n)$ is given by

$$(12) \qquad \left\{ \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & E_{ij} \\ E_{ij}^\mathsf{T} & 0 \end{bmatrix} \in \mathbb{S}^n : i = 1,\ldots,k,\ j = 1,\ldots,n-k \right\}$$

where E_{ij} is the $k \times (n-k)$ matrix with one in the (i,j)th entry and zero everywhere else. We refer readers to [27, Section 3] for the proofs of statements in this and the last paragraph.

The next two results require no calculation and are just stated for completeness.

Proposition 4.1 (First fundamental form). *The first fundamental form* $\mathrm{I}_Q : \mathbb{T}_Q\,\mathrm{Gr}(k,n) \times \mathbb{T}_Q\,\mathrm{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathrm{I}_Q(X,Y) = 2\,\mathrm{tr}(X_0^\mathsf{T}Y_0)$$

with Q, X, Y parameterized as in (11).

Proposition 4.2 (Gauss map). *The Gauss map of* $\mathrm{Gr}(k,n)$ *in* $\mathbb{S}^n$ *is given by*

$$\Gamma : \mathrm{Gr}(k,n) \to \mathrm{Gr}\!\left(\binom{n+1}{2} - k(n-k), \mathbb{S}^n\right),$$

$$\Gamma(Q) = \mathbb{N}_Q(\mathrm{Gr}(k,n)) = V \left\{ \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} : H_1 \in \mathbb{S}^k, H_2 \in \mathbb{S}^{n-k} \right\} V^\mathsf{T},$$

with Q parameterized as in (11).

The next calculation is our key to unlocking other calculations in this article.

Theorem 4.3 (Second fundamental form)**.** *The second fundamental form* $\mathrm{II}_Q : \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \to \mathbb{N}_Q \operatorname{Gr}(k,n)$ *is given by*

$$\mathrm{II}_Q(X,Y) = \frac{1}{2}V\begin{bmatrix} X_0 Y_0^\mathsf{T} + Y_0 X_0^\mathsf{T} & 0 \\ 0 & -X_0^\mathsf{T} Y_0 - Y_0^\mathsf{T} X_0 \end{bmatrix} V^\mathsf{T},$$

with Q, X, Y *parameterized as in* (11).

Proof. Since $\mathrm{O}(n)$ acts on $\operatorname{Gr}(k,n)$ transitively and isometrically, it suffices to calculate II at $I_{k,n-k} \in \operatorname{Gr}(k,n)$. In this case $X,Y \in \mathbb{T}_{I_{k,n-k}} \operatorname{Gr}(k,n)$ may be written as

$$X = \begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & Y_0 \\ Y_0^\mathsf{T} & 0 \end{bmatrix}$$

for some $X_0, Y_0 \in \mathbb{R}^{k \times (n-k)}$. Points near $I_{k,n-k}$ can be parametrized as

$$\varphi(B, H_1, H_2) = \exp\left(\begin{bmatrix} 0 & -B \\ B^\mathsf{T} & 0 \end{bmatrix}\right)\left(I_{k,n-k} + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}\right)\exp\left(\begin{bmatrix} 0 & -B \\ B^\mathsf{T} & 0 \end{bmatrix}^\mathsf{T}\right),$$

where $B \in \mathbb{R}^{k \times (n-k)}$, $H_1 \in \mathbb{S}^k$, and $H_2 \in \mathbb{S}^{n-k}$ have sufficiently small norms. Clearly, we have $\varphi(B, H_1, H_2) \in \operatorname{Gr}(k,n)$ if and only if $H_1 = 0$ and $H_2 = 0$. Thus we may extend X by

$$\widetilde{X}\big(\varphi(B, H_1, H_2)\big) = \exp\left(\begin{bmatrix} 0 & -B \\ B^\mathsf{T} & 0 \end{bmatrix}\right)\begin{bmatrix} H_1 & X_0 \\ X_0^\mathsf{T} & H_2 \end{bmatrix}\exp\left(\begin{bmatrix} 0 & -B \\ B^\mathsf{T} & 0 \end{bmatrix}\right)^\mathsf{T}.$$

Such an $\widetilde{X}$ is an extension of a local vector field around $I_{k,n-k}$ on $\operatorname{Gr}(k,n)$. By (5),

$$\mathrm{II}_{I_{k,n-k}}(X,Y) = \operatorname{proj}_{\mathbb{N}_{I_{k,n-k}} \operatorname{Gr}(k,n)}\big(\langle \widetilde{\nabla}\widetilde{X}(I_{k,n-k}), Y \rangle\big)$$

where $\widetilde{\nabla}$ denotes the covariant derivative in the Euclidean space $\mathbb{S}^n$, i.e.,

$$\widetilde{\nabla}\widetilde{X} = \big(\partial_B \widetilde{X}, \partial_{H_1} \widetilde{X}, \partial_{H_2} \widetilde{X}\big)$$

Since Y is a tangent vector, we obtain

$$\langle \widetilde{\nabla}\widetilde{X}(I_{k,n-k}), Y \rangle = \sum_{i=1}^{k}\sum_{j=1}^{n-k} \frac{\partial \widetilde{X}}{\partial b_{ij}}(I_{k,n-k}) y_{0ij},$$

where we have written $B = (b_{ij})$ and $Y_0 = (y_{0ij})$. Observe that

$$\frac{\partial \widetilde{X}}{\partial b_{ij}}(I_{k,n-k}) = -\frac{1}{2}\begin{bmatrix} 0 & E_{ij} \\ -E_{ij}^\mathsf{T} & 0 \end{bmatrix}\begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix}\begin{bmatrix} 0 & E_{ij} \\ -E_{ij}^\mathsf{T} & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -E_{ij}X_0^\mathsf{T} - X_0 E_{ij}^\mathsf{T} & 0 \\ 0 & E_{ij}^\mathsf{T} X_0 + X_0^\mathsf{T} E_{ij} \end{bmatrix}$$

where the factor $\frac{1}{2}$ is a result of our choice of Riemannian metric on $\mathbb{S}^n$. Therefore we have

$$\sum_{i=1}^{k}\sum_{j=1}^{n-k} \frac{\partial \widetilde{X}}{\partial b_{ij}}(I_{k,n-k}) y_{0ij} = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n-k}\begin{bmatrix} -(E_{ij}X_0^\mathsf{T} + X_0 E_{ij}^\mathsf{T})y_{0ij} & 0 \\ 0 & (E_{ij}^\mathsf{T} X_0 + X_0^\mathsf{T} E_{ij})y_{0ij} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -X_0 Y_0^\mathsf{T} - Y_0 X_0^\mathsf{T} & 0 \\ 0 & X_0^\mathsf{T} Y_0 + Y_0^\mathsf{T} X_0 \end{bmatrix},$$

where the last expression is our required $\mathrm{II}_{I_{k,n-k}}(X,Y)$. $\square$

We record an observation that follows from an additional step of singular value decomposition.

Corollary 4.4 (Index of relative nullity)**.** *The index of relative nullity* ν_Q *of* $\operatorname{Gr}(k,n)$ *is zero.*

Proof. Let $X \in \mathbb{T}_Q \operatorname{Gr}(k, n)$ be such that $\mathrm{II}_Q(X, Y) = 0$ for all $Y \in \mathbb{T}_Q \operatorname{Gr}(k, n)$, with Q, X, Y parametrized as in (11). We claim that $X = 0$. By Theorem 4.3, we must have

$$(13) \qquad X_0 Y_0^\mathsf{T} + Y_0 X_0^\mathsf{T} = 0, \quad X_0^\mathsf{T} Y_0 + Y_0^\mathsf{T} X_0 = 0$$

for any $Y_0 \in \mathbb{R}^{k \times (n-k)}$. Let $X_0 = U \Sigma V^\mathsf{T}$ be a singular value decomposition with $U \in \mathrm{O}(k)$ and $V \in \mathrm{O}(n-k)$. Then (13) becomes

$$\Sigma (U^\mathsf{T} Y_0 V)^\mathsf{T} + (U^\mathsf{T} Y_0 V)\Sigma = 0, \quad \Sigma(U^\mathsf{T} Y_0 V) + (U^\mathsf{T} Y_0 V)^\mathsf{T} \Sigma = 0.$$

Since Y_0 is arbitrary, we may set $X_0 = \Sigma$ in (13). Now by taking Y_0 to be an arbitrary diagonal $k \times (n-k)$ matrix, we see that $\Sigma = 0$. Hence $X_0 = 0$ and $X = 0$. $\qquad\square$

The Weingarten map is an alternative way to express the second fundamental form and thus follows easily from Theorem 4.3.

Corollary 4.5 (Weingarten map). *The Weingarten map $\mathsf{S}_Q(H) : \mathbb{T}_Q \operatorname{Gr}(k, n) \to \mathbb{T}_Q \operatorname{Gr}(k, n)$ along the normal direction $H \in \mathbb{N}_Q \operatorname{Gr}(k, n)$ is given by*

$$(14) \qquad \mathsf{S}_Q(H)(X) = \frac{1}{2} V \begin{bmatrix} 0 & H_1 X_0 - X_0 H_2 \\ (H_1 X_0 - X_0 H_2)^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}$$

with Q, X, H parameterized as in (11).

Proof. We plug in the expressions from (11) into $\langle \mathsf{S}_Q(H)(X), Y \rangle = \langle \mathrm{II}_Q(X, Y), H \rangle$ and use standard properties of trace to get

$$\langle \mathsf{S}_Q(H)(X), Y \rangle = \frac{1}{2} \left[\operatorname{tr}\big((X_0 Y_0^\mathsf{T} + Y_0 X_0^\mathsf{T}) H_1\big) - \operatorname{tr}\big((X_0^\mathsf{T} Y_0 + Y_0^\mathsf{T} X_0) H_2\big) \right]$$

$$= \frac{1}{2} \left[\operatorname{tr}\big((H_1 X_0 - X_0 H_2)) Y_0^\mathsf{T}\big) + \operatorname{tr}\big((X_0^\mathsf{T} H_1 - H_2 X_0^\mathsf{T}) Y_0\big) \right]$$

$$= \operatorname{tr}\left(\frac{1}{2} \begin{bmatrix} 0 & H_1 X_0 - X_0 H_2 \\ X_0^\mathsf{T} H_1 - H_2 X_0^\mathsf{T} & 0 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & Y_0 \\ Y_0^\mathsf{T} & 0 \end{bmatrix} \right),$$

and thereby deducing (14). $\qquad\square$

The calculation of mean curvature is also straightforward.

Corollary 4.6 (Mean curvature). *The mean curvature vector of $\operatorname{Gr}(k, n)$ is given by*

$$\mathsf{H}_Q = \frac{1}{k(n-k)} \operatorname{tr}(\mathrm{II}_Q) = \frac{1}{2k(n-k)} V \begin{bmatrix} -(n-k) I_k & 0 \\ 0 & k I_{n-k} \end{bmatrix} V^\mathsf{T}$$

and the mean curvature of $\operatorname{Gr}(k, n)$ along $H \in \mathbb{N}_Q \operatorname{Gr}(k, n)$ is given by

$$\mathsf{H}_Q(H) = \frac{(k-n) \operatorname{tr} H_1 + k \operatorname{tr} H_2}{2k(n-k)}$$

with Q, X, H parameterized as in (11).

Proof. We use the orthonormal basis of $\mathbb{T}_{I_{k,n-k}} \operatorname{Gr}(k, n)$ in (12). A straightforward but slightly messy calculation gives

$$(E_{ij} E_{i'j'}^\mathsf{T})_{pq} = \begin{cases} 1 & \text{if } j = j' \text{ and } (p, q) = (i, i'), \\ 0 & \text{otherwise}, \end{cases}$$

for any $p, q \in \{1, \ldots, k\}$; and

$$(E_{ij}^\mathsf{T} E_{i'j'})_{pq} = \begin{cases} 1 & \text{if } i = i' \text{ and } (p, q) = (j, j') \\ 0 & \text{otherwise} \end{cases}$$

for any $p, q \in \{1, \ldots, n - k\}$. Using these, we may evaluate

$$\mathrm{II}_{I_{k,n-k}}\left(\begin{bmatrix} 0 & E_{ij} \\ E_{ij}^\mathsf{T} & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_{i'j'} \\ E_{i'j'}^\mathsf{T} & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -\delta_{jj'}(E_{ii'} + E_{i'i}) & 0 \\ 0 & \delta_{ii'}(E_{jj'} + E_{j'j}) \end{bmatrix}$$

and obtain the required expression by summing over the basis. The mean curvature along H is then calculated from $\mathsf{H}_Q(H) = \langle \mathsf{H}_Q, H \rangle$.   □

Corollary 4.7 (Principal and Gaussian curvatures). *Let $Q \in \mathrm{Gr}(k, n)$ and $H \in \mathbb{N}_Q \mathrm{Gr}(k, n)$ be parameterized as in* (11). *Then the Weingarten map $\mathsf{S}_Q(H)$ has eigenpairs given by*

$$\left(\frac{1}{2}(\lambda_{k+j} - \lambda_i), V \begin{bmatrix} 0 & Q_1 E_{ij} Q_2^\mathsf{T} \\ Q_2 E_{ij}^\mathsf{T} Q_1^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}\right), \quad i = 1, \ldots, k, \; j = 1, \ldots, n - k,$$

where $H_1 = Q_1 \Lambda_1 Q_1^\mathsf{T}$ and $H_2 = Q_2 \Lambda_2 Q_2^\mathsf{T}$ are eigenvalue decompositions with $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ and $\Lambda_2 = \mathrm{diag}(\lambda_{k+1}, \ldots, \lambda_n)$.

(a) *The principal curvatures of $\mathrm{Gr}(k, n)$ along H are*

$$\kappa_{ij} = \frac{1}{2}(\lambda_{k+j} - \lambda_i), \quad i = 1, \ldots, k, \; j = 1, \ldots, n - k.$$

(b) *The Gaussian curvature of $\mathrm{Gr}(k, n)$ along H is*

$$\mathsf{G}_Q(H) = \frac{1}{2^{k(n-k)}} \prod_{i=1}^{k} \prod_{j=1}^{n-k} (\lambda_{k+j} - \lambda_i).$$

Proof. By (14), we have

$$\mathsf{S}_Q(H)\left(V \begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}\right) = V\mathsf{S}_{I_{k,n-k}}(H_0)\left(\begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix}\right) V^\mathsf{T}, \quad H_0 := \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}.$$

Write $\Lambda := \mathrm{diag}(\Lambda_1, \Lambda_2) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$\begin{aligned}
\mathsf{S}_{I_{k,n-k}}(H_0)\left(\begin{bmatrix} 0 & X_0 \\ X_0^\mathsf{T} & 0 \end{bmatrix}\right) &= \frac{1}{2}\begin{bmatrix} 0 & H_1 X_0 - X_0 H_2 \\ (H_1 X_0 - X_0 H_2)^\mathsf{T} & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & Q_1 \Lambda_1 Q_1^\mathsf{T} X_0 - X_0 Q_2 \Lambda_2 Q_2^\mathsf{T} \\ (Q_1 \Lambda_1 Q_1^\mathsf{T} X_0 - X_0 Q_2 \Lambda_2 Q_2^\mathsf{T})^\mathsf{T} & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} 0 & Y_0 \\ Y_0^\mathsf{T} & 0 \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}^\mathsf{T} \\
&= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\mathsf{S}_{I_{k,n-k}}(\Lambda)\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}^\mathsf{T},
\end{aligned}$$

where $Y_0 := \Lambda_1(Q_1^\mathsf{T} X_0 Q_2) - (Q_1^\mathsf{T} X_0 Q_2)\Lambda_2$. So it suffices to diagonalize the linear operator $\mathsf{S}_{I_{k,n-k}}(\Lambda) : \mathbb{T}_{I_{k,n-k}} \mathrm{Gr}(k, n) \to \mathbb{T}_{I_{k,n-k}} \mathrm{Gr}(k, n)$. Now observe that

$$\mathsf{S}_{I_{k,n-k}}(\Lambda)\left(\frac{\sqrt{2}}{2}\begin{bmatrix} 0 & E_{ij} \\ E_{ij}^\mathsf{T} & 0 \end{bmatrix}\right) = \delta_{ip}\delta_{jq}\frac{\lambda_{k+j} - \lambda_i}{2}\left(\frac{\sqrt{2}}{2}\begin{bmatrix} 0 & E_{pq} \\ E_{pq}^\mathsf{T} & 0 \end{bmatrix}\right)$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, n - k$, gives us the required diagonalization, which is an eigenvalue decomposition as (12) is an orthonormal basis. The values of the principal and Gaussian curvatures follow.   □

The easiest way to calculate the third fundamental form is to get slightly ahead of our discussion and use the expression for Ricci curvature in Corollary 5.4 together with a result of Obata [37, Theorem 1]. Otherwise we would have to start from the definition in Section 3.1.

Corollary 4.8 (Third fundamental form). *The third fundamental form* $\mathrm{III}_Q : \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathrm{III}_Q(X,Y) = -\frac{1}{2}\Big(\frac{n}{2k(n-k)} + \frac{n-2}{4}\Big)\operatorname{tr}(XY) = -\Big(\frac{n}{2k(n-k)} + \frac{n-2}{4}\Big)\operatorname{tr}(X_0^\mathsf{T} Y_0),$$

with Q, X, Y *parameterized as in* (11).

Proof. By [37, Theorem 1], we have

$$\mathrm{III}_Q(X,Y) = \langle \mathrm{II}_Q(X,Y), \mathsf{H}_Q\rangle - \mathsf{Ric}(X,Y).$$

By Theorem 4.3, Corollaries 4.6 and 5.4, we have

$$\mathrm{III}_Q(X,Y) = \frac{1}{4k(n-k)}\big(-(n-k)\operatorname{tr}(X_0 Y_0^\mathsf{T} + Y_0 X_0^\mathsf{T}) - k\operatorname{tr}(X_0^\mathsf{T} Y_0 + Y_0^\mathsf{T} X_0)\big) - \frac{n-2}{4}\operatorname{tr}(X_0^\mathsf{T} Y_0)$$

$$= -\Big(\frac{n}{2k(n-k)} + \frac{n-2}{4}\Big)\operatorname{tr}(X_0^\mathsf{T} Y_0). \qquad \square$$

5. Intrinsic curvatures of the Grassmannian

As we will see in Section 7.2, calculating intrinsic curvatures of Grassmannian with intrinsic geometry can get fairly involved. This is particularly striking for the Riemann curvature tensor — our calculation below is essentially one-line using the embedded geometry of the involution model.

Proposition 5.1 (Riemmanian curvature). *The Riemann tensor* $\mathsf{Rie}_Q : \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathsf{Rie}_Q(X,Y,Z,W) = \frac{1}{2}\operatorname{tr}\big((XY - YX)ZW\big)$$

$$= \frac{1}{2}\operatorname{tr}\big((X_0^\mathsf{T} Y_0 Z_0^\mathsf{T} + Z_0^\mathsf{T} Y_0 X_0^\mathsf{T} - Y_0^\mathsf{T} X_0 Z_0^\mathsf{T} - Z_0^\mathsf{T} X_0 Y_0^\mathsf{T})W_0\big)$$

with Q, X, Y, Z, W *parameterized as in* (11).

Proof. Using the expression for II_Q,

$$\mathsf{Rie}_Q(X,Y,Z,W) = \langle \mathrm{II}_Q(Y,Z), \mathrm{II}_Q(X,W)\rangle - \langle \mathrm{II}_Q(X,Z), \mathrm{II}_Q(Y,W)\rangle$$

$$= \frac{1}{4}\langle Y_0 Z_0^\mathsf{T} + Z_0 Y_0^\mathsf{T}, X_0 W_0^\mathsf{T} + W_0 X_0^\mathsf{T}\rangle + \frac{1}{4}\langle Y_0^\mathsf{T} Z_0 + Z_0^\mathsf{T} Y_0, X_0^\mathsf{T} W_0 + W_0^\mathsf{T} X_0\rangle$$

$$\quad - \frac{1}{4}\langle X_0 Z_0^\mathsf{T} + Z_0 X_0^\mathsf{T}, Y_0 W_0^\mathsf{T} + W_0 Y_0^\mathsf{T}\rangle - \frac{1}{4}\langle X_0^\mathsf{T} Z_0 + Z_0^\mathsf{T} X_0, Y_0^\mathsf{T} W_0 + W_0^\mathsf{T} Y_0\rangle$$

$$= \frac{1}{4}\langle [[X,Y],Z], W\rangle$$

$$= \frac{1}{2}\operatorname{tr}\big((XY - YX)ZW\big),$$

where the last equality is obtained by observing that X, Y, Z, W are symmetric matrices. $\qquad \square$

Corollary 5.2 (Jacobi curvature). *The Jacobi tensor* $\mathsf{J}_Q : \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \to \mathbb{R}$ *is*

$$\mathsf{J}_Q(X,Y,Z,W) = \operatorname{tr}(XYZW) - \operatorname{tr}\Big(Y\Big(\frac{XZ+ZX}{2}\Big)W\Big)$$

$$= \operatorname{tr}\big((X_0^\mathsf{T} Y_0 Z_0^\mathsf{T} + Z_0^\mathsf{T} Y_0 X_0^\mathsf{T})W_0\big)$$

$$\quad - \operatorname{tr}\Big(Y_0^\mathsf{T}\Big(\frac{X_0 Z_0^\mathsf{T} + Z_0 X_0^\mathsf{T}}{2}\Big)W_0\Big) - \operatorname{tr}\Big(\Big(\frac{Z_0^\mathsf{T} X_0 + X_0^\mathsf{T} Z_0}{2}\Big)Y_0^\mathsf{T} W_0\Big),$$

with Q, X, Y, Z, W *parameterized as in* (11).

Proof. The expression in Proposition 5.1 and the fact that X, Y, Z, W are symmetric matrices yield

$$\mathsf{J}_Q(X, Y, Z, W) = \frac{1}{2} \operatorname{tr}\big((2XYZ - Y(XZ + ZX)W\big)$$

and thus the first expression. Plugging in the parameterizations in (11) for X, Y, Z, W gives the second expression. $\qquad\square$

Corollary 5.3 (Sectional curvature)**.** *The sectional curvature* $\kappa_Q : \mathbb{T}_Q \operatorname{Gr}(k, n) \times \mathbb{T}_Q \operatorname{Gr}(k, n) \to \mathbb{R}$ *is given by*

$$\kappa_Q(X, Y) = \frac{\|[X, Y]\|^2}{4(\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2)} = \frac{\|[X_0, Y_0^\mathsf{T}]\|^2 + \|[X_0^\mathsf{T}, Y_0]\|^2}{16(\|X_0\|^2 \|Y_0\|^2 - \langle X_0, Y_0\rangle^2)} \le \frac{1}{4},$$

with Q, X, Y *parameterized as in* (11)*. If* X, Y *are orthonormal, i.e.,* $\|X_0\| = \|Y_0\| = \sqrt{2}/2$ *and* $\langle X_0, Y_0\rangle = 0$*, then*

$$\kappa_Q(X, Y) = \frac{\|[X, Y]\|^2}{4} = \frac{1}{4}(\|[X_0, Y_0^\mathsf{T}]\|^2 + \|[X_0^\mathsf{T}, Y_0]\|^2).$$

Proof. This is a straightforward calculation by

$$\kappa_Q(X, Y) = \frac{\mathsf{Rie}_Q(X, Y, Y, X)}{\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2} = \frac{1}{4} \frac{\langle [[X, Y], Y], X\rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2} = \frac{\|[X, Y]\|^2}{4(\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2)}$$

and the observations that

$$[X, Y] = V\left(\begin{bmatrix} X_0 Y_0^\mathsf{T} - Y_0 X_0^\mathsf{T} & 0 \\ 0 & X_0^\mathsf{T} Y_0 - Y_0^\mathsf{T} X_0 \end{bmatrix}\right) V^\mathsf{T},$$

$$\|X\|^2 = 2\|X_0\|^2, \quad \|Y\|^2 = 2\|Y_0\|^2, \quad \|\langle X, Y\rangle\|^2 = 2\langle X_0, Y_0\rangle.$$

Since $\kappa_Q(X, Y)$ only depends on the two-dimensional subspace of $\mathbb{T}_Q \operatorname{Gr}(k, n)$ spanned by X and Y, it suffices to assume that X, Y are orthonormal. The upper bound $\kappa_Q(X, Y) \le 1/4$ then follows from the inequality $\|[A, B]\|^2 \le 2\|A\|^2 \|B\|^2$ for any $A, B \in \mathbb{R}^{n \times n}$. $\qquad\square$

It is known that the Grassmannian manifolds are Einstein [5, Paragraphs 0.25 and 0.26]. Our calculations below confirm the fact.

Corollary 5.4 (Ricci and scalar curvatures)**.** *Let* Q, X, Y *be parameterized as in* (11)*. The Ricci tensor* $\mathsf{Ric}_Q : \mathbb{T}_Q \operatorname{Gr}(k, n) \times \mathbb{T}_Q \operatorname{Gr}(k, n) \to \mathbb{R}$ *is given by*

$$\mathsf{Ric}_Q(X, Y) = \frac{(n-2)}{8} \operatorname{tr}(XY) = \frac{(n-2)}{4} \operatorname{tr}(X_0^\mathsf{T} Y_0).$$

The scalar curvature $\mathsf{Sca}_Q \in \mathbb{R}$ *is given by*

$$\mathsf{Sca}_Q = \frac{k(n-k)(n-2)}{8}.$$

The traceless Ricci curvature $\mathsf{Z}_Q : \mathbb{T}_Q \operatorname{Gr}(k, n) \times \mathbb{T}_Q \operatorname{Gr}(k, n) \to \mathbb{R}$ *is given by*

$$\mathsf{Z}_Q(X, Y) = 0,$$

which shows that the Grassmannian is an Einstein manifold.

Proof. We write

$$X_{ij} := \frac{\sqrt{2}}{2} V \begin{bmatrix} 0 & E_{ij} \\ E_{ij}^\mathsf{T} & 0 \end{bmatrix} V^\mathsf{T}$$

for the elements of the orthonormal basis in (12). Then

$$\mathsf{Ric}_Q(X,Y) = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\mathsf{Rie}_Q(X_{ij},X,Y,X_{ij}) = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\frac{1}{4}\langle[[X_{ij},X],Y],X_{ij}\rangle\rangle$$

$$= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n-k}\mathrm{tr}(X_{ij}^2 XY - XX_{ij}YX_{ij})$$

$$= \frac{n-2}{8}\mathrm{tr}(XY) = \frac{\mathsf{Sca}_Q}{k(n-k)}\mathsf{g}_Q$$

where the last equality shows that Z_Q vanishes identically. $\qquad\square$

For a homogeneous space like $\mathrm{Gr}(k,n)$, the upper and lower delta invariants $\overline{\delta}_Q(d_1,\ldots,d_r)$ and $\underline{\delta}_Q(d_1,\ldots,d_r)$ are independent of the choice of $Q \in \mathrm{Gr}(k,n)$. We also restrict our attention to $d_1 = \cdots = d_r = 2$. So for notational simplicity, we just write

$$\overline{\delta}_{2,r} := \overline{\delta}_Q(\underbrace{2,\ldots,2}_{r \text{ times}}), \quad \underline{\delta}_{2,r} := \underline{\delta}_Q(\underbrace{2,\ldots,2}_{r \text{ times}}).$$

Theorem 5.5 (Delta invariants). *Let $r \leq 2\lfloor k/2\rfloor\lfloor(n-k)/2\rfloor$. Then the upper and lower delta invariants of $\mathrm{Gr}(k,n)$ are given by*

$$\overline{\delta}_{2,r} = \frac{k(n-k)(n-2)}{8}, \qquad \underline{\delta}_{2,r} = \frac{k(n-k)(n-2)}{8} - \frac{r}{4}.$$

Proof. We will write $\kappa = \kappa_Q$ below and take $Q = I_{k,n-k}$. By Corollary 5.4, we have

$$\overline{\delta}_{2,r} = \frac{k(n-k)(n-2)}{8} - \inf_{\substack{\dim \mathbb{V}_j=2,\\ \mathbb{V}_j\perp\mathbb{V}_k,j<k}}\left[\sum_{j=1}^{r}\kappa(X_j,Y_j)\right],$$

$$\underline{\delta}_{2,r} = \frac{k(n-k)(n-2)}{8} - \sup_{\substack{\dim \mathbb{V}_j=2,\\ \mathbb{V}_j\perp\mathbb{V}_k,j<k}}\left[\sum_{j=1}^{r}\kappa(X_j,Y_j)\right],$$

where $\{X_j,Y_j\}$ is an orthonormal basis of the two-dimensional subspace $\mathbb{V}_j \subseteq \mathbb{T}_{I_{k,n-k}}\mathrm{Gr}(k,n)$, $j = 1,\ldots,r$. By Corollary 5.3, we have

(15)
$$0 \leq \sum_{j=1}^{r}\kappa(X_j,Y_j) \leq \frac{r}{4}.$$

It remains to show that upper and lower bounds in (15) are attained by some $\mathbb{V}_1,\ldots,\mathbb{V}_r$.

Set $k_1 := \lfloor k/2\rfloor$ and $k_2 := \lfloor(n-k)/2\rfloor$. We may partition any $X \in \mathbb{T}_{I_{k,n-k}}\mathrm{Gr}(k,n)$ into a block matrix with 2×2 blocks $B_{pq} \in \mathbb{R}^{2\times 2}$:

$$X = \begin{bmatrix} 0 & \cdots & 0 & B_{1,1} & \cdots & B_{1,k_2+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{k_1+1,1} & \cdots & B_{k_1+1,k_2+1} \\ B_{1,1}^\mathsf{T} & \cdots & B_{k_1+1,1}^\mathsf{T} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{1,k_2+1}^\mathsf{T} & \cdots & B_{k_1+1,k_2+1}^\mathsf{T} & 0 & \cdots & 0 \end{bmatrix}$$

except in the last row and column where we are required to have

$$B_{p,k_2+1} \in \mathbb{R}^{2\times(n-k-2k_2)}, \quad B_{k_1+1,q} \in \mathbb{R}^{(k-2k_1)\times 2}, \quad B_{k_1+1,k_2+1} \in \mathbb{R}^{(k-2k_1)\times(n-k-2k_2)}$$

for $p = 1,\ldots,k_1$ and $q = 1,\ldots,k_2$.

Let $\widehat{X}_{ij} \in \mathbb{T}_{I_{k,n-k}} \mathrm{Gr}(k,n)$ be the tangent vector obtained from X by setting $B_{pq} = 0$ whenever $(p,q) \neq (i,j)$. Then clearly we have $\mathrm{tr}\big(\widehat{X}_{ij}^{\mathsf{T}}\widehat{X}_{i'j'}\big) = 0$ whenever $(i,j) \neq (i',j')$. Since $r \leq 2k_1 k_2$, the problem further reduces to attaining the upper and lower bounds in (15) for $r = 2$ on $\mathrm{Gr}(2,4)$. This is a vast simplification as $X \in \mathbb{T}_{I_{2,2}} \mathrm{Gr}(2,4)$ is just $X = \begin{bmatrix} 0 & B \\ B^{\mathsf{T}} & 0 \end{bmatrix}$ with $B \in \mathbb{R}^{2\times2}$. It remains to exhibit an orthonormal basis $X_1, Y_1, X_2, Y_2 \in T_{I_{2,2}} \mathrm{Gr}(2,4)$ that gives the upper and lower bounds in (15). Using the formula for sectional curvature in Corollary 5.3, we check that

$$X_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad Y_1 = \frac{1}{2} \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}, \quad X_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&1&0&0 \end{bmatrix}, \quad Y_2 = \frac{1}{2} \begin{bmatrix} 0&0&0&1 \\ 0&0&-1&0 \\ 0&-1&0&0 \\ 1&0&0&0 \end{bmatrix}$$

give the required upper bound $\kappa(X_1, Y_1) + \kappa(X_2, Y_2) = \frac{1}{2}$, whereas

$$X_1 = \frac{1}{2} \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix}, \quad Y_1 = \frac{1}{2} \begin{bmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{bmatrix}, \quad X_2 = \frac{1}{2} \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}, \quad Y_2 = \frac{1}{2} \begin{bmatrix} 0&0&0&1 \\ 0&0&-1&0 \\ 0&-1&0&0 \\ 1&0&0&0 \end{bmatrix}$$

give the required lower bound $\kappa(X_1, Y_1) + \kappa(X_2, Y_2) = 0$. $\qquad\square$

We next compute the quartet of tensors named after Schouten, Cotton, Weyl, and Bach.

Corollary 5.6 (Schouten curvature). *The Schouten tensor* $\mathsf{P}_Q : \mathbb{T}_Q \mathrm{Gr}(k,n) \times \mathbb{T}_Q \mathrm{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathsf{P}_Q(X,Y) = \frac{(n-2)}{16(k(n-k)-1)} \mathrm{tr}(XY) = \frac{2(n-2)}{16(k(n-k)-1)} \mathrm{tr}(X_0^{\mathsf{T}} Y_0)$$

with Q, X, Y *parameterized as in* (11).

Proof. This is a straightforward calculation from definition:

$$\mathsf{P}_Q(X,Y) = \frac{1}{k(n-k)-2}\left[\mathsf{Ric}_Q(X,Y) - \frac{\mathsf{Sca}_Q}{2(k(n-k)-1)} \mathsf{g}_Q(X,Y) \right]$$
$$= \frac{n-2}{16(k(n-k)-1)} \mathrm{tr}(XY). \qquad\square$$

Corollary 5.7 (Cotton curvature). *The Cotton tensor of* $\mathrm{Gr}(k,n)$ *is zero.*

Proof. By Corollary 5.6, P is a constant multiple of g. So $\nabla\mathsf{P} = 0$, and so C is identically zero. $\square$

Corollary 5.8 (Weyl curvature). *The Weyl tensor* $\mathsf{W}_Q : \mathbb{T}_Q \mathrm{Gr}(k,n) \times \mathbb{T}_Q \mathrm{Gr}(k,n) \times \mathbb{T}_Q \mathrm{Gr}(k,n) \times \mathbb{T}_Q \mathrm{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathsf{W}_Q(X,Y,Z,W) = \frac{1}{2} \mathrm{tr}\big((XY - YX)ZW\big)$$
$$- \frac{(n-2)}{8(k(n-k)-1)}\big(\mathrm{tr}(XZ)\,\mathrm{tr}(YW) - \mathrm{tr}(XW)\,\mathrm{tr}(YZ)\big)$$
$$= \frac{1}{2} \mathrm{tr}\big((X_0^{\mathsf{T}} Y_0 Z_0^{\mathsf{T}} + Z_0^{\mathsf{T}} Y_0 X_0^{\mathsf{T}} - Y_0^{\mathsf{T}} X_0 Z_0^{\mathsf{T}} - Z_0^{\mathsf{T}} X_0 Y_0^{\mathsf{T}})W_0\big)$$
$$- \frac{n-2}{2(k(n-k)-1)}\big(\mathrm{tr}(X_0^{\mathsf{T}} Z_0)\,\mathrm{tr}(Y_0^{\mathsf{T}} W_0) - \mathrm{tr}(X_0^{\mathsf{T}} W_0)\,\mathrm{tr}(Y_0^{\mathsf{T}} Z_0)\big)$$

with Q, X, Y, Z, W *parameterized as in* (11).

Proof. Let $m := k(n-k)$. It follows from the vanishing of Z_Q and the expression for Sca_Q in Corollary 5.4 that

$$\mathsf{W}_Q = \mathsf{Rie}_Q - \frac{1}{m-2}\mathsf{Z}_Q \varowedge \mathsf{g}_Q - \frac{\mathsf{Sca}_Q}{2m(m-1)}\mathsf{g}_Q \varowedge \mathsf{g}_Q = \mathsf{Rie}_Q - \frac{n-2}{16(m-1)}\mathsf{g}_Q \varowedge \mathsf{g}_Q.$$

Next use the two expressions of Rie_Q in Proposition 5.1 and expand the Kulkarni–Nomizu product

$$\begin{aligned}
\mathsf{g}_Q \owedge \mathsf{g}_Q(X,Y,Z,W) &= \mathsf{g}_Q(X,Z)\mathsf{g}_Q(Y,W) - \mathsf{g}_Q(X,W)\mathsf{g}_Q(Y,Z) \\
&\quad - \mathsf{g}_Q(Y,Z)\mathsf{g}_Q(X,W) + \mathsf{g}_Q(Y,W)\mathsf{g}_Q(X,Z) \\
&= 2\big(\mathsf{g}_Q(X,Z)\mathsf{g}_Q(Y,W) - \mathsf{g}_Q(X,W)\mathsf{g}_Q(Y,Z)\big) \\
&= 8\big(\operatorname{tr}(X_0^\mathsf{T} Z_0)\operatorname{tr}(Y_0^\mathsf{T} W_0) - \operatorname{tr}(X_0^\mathsf{T} W_0)\operatorname{tr}(Y_0^\mathsf{T} Z_0)\big),
\end{aligned}$$

to get the two required expressions for $\mathsf{W}_Q(X,Y,Z,W)$. $\qquad\square$

Corollary 5.9 (Bach curvature). *The Bach tensor* $\mathsf{B}_Q : \mathbb{T}_Q \operatorname{Gr}(k,n) \times \mathbb{T}_Q \operatorname{Gr}(k,n) \to \mathbb{R}$ *is given by*

$$\mathsf{B}_Q(X,Y) = \frac{(n-2)^2}{32(k(n-k)-2)}\operatorname{tr}(XY) = \frac{(n-2)^2}{16(k(n-k)-2)}\operatorname{tr}(X_0^\mathsf{T} Y_0)$$

with Q, X, Y *parameterized as in* (11).

Proof. Let $m := k(n-k)$. Using [12, Equation (2-4)], we relate B to the Cotton tensor C as

$$\mathsf{B}_Q(X,Y) = \frac{1}{m-2}\left[\sum_{i=1}^m (\nabla_{X_i}\mathsf{C}_Q)(X_i,X,Y) + \sum_{i=1}^m \sum_{j=1}^m \mathsf{Ric}_Q(X_i,X_j)\mathsf{W}_Q(X,X_i,X_j,Y)\right],$$

where $X_1,\ldots,X_m \in \mathbb{T}_Q \operatorname{Gr}(k,n)$ is any orthonormal basis. It follows from the vanishing of C in Corollary 5.7 that

$$\begin{aligned}
\mathsf{B}_Q(X,Y) &= \frac{n-2}{8(m-2)}\sum_{i=1}^m \mathsf{W}_Q(X,X_i,X_i,Y) \\
&= \frac{n-2}{8(m-2)}\left[\sum_{i=1}^m \mathsf{Rie}_Q(X,X_i,X_i,Y) - \sum_{i=1}^m \frac{n-2}{16(m-1)}\mathsf{g}_Q \owedge \mathsf{g}_Q(X,X_i,X_i,Y)\right] \\
&= \frac{n-2}{8(m-2)}\left[\mathsf{Ric}_Q(X,Y) - \frac{2(n-2)}{16(m-1)}\sum_{i=1}^m (\mathsf{g}_Q(X,X_i)\mathsf{g}_Q(Y,X_i) - \mathsf{g}_Q(X,Y)\mathsf{g}_Q(X_i,X_i))\right] \\
&= \frac{n-2}{8(m-2)}\mathsf{Ric}_Q(X,Y) - \frac{(n-2)^2}{64(m-1)(m-2)}(1-m)\mathsf{g}_Q(X,Y) \\
&= \frac{(n-2)^2}{32(m-2)}\mathsf{g}_Q(X,Y).
\end{aligned}$$

Here the first and the second equalities follow from Corollaries 5.4 and 5.8 respectively. The third uses the definition of Ricci curvature and the value $\mathsf{g}_Q \owedge \mathsf{g}_Q$ calculated in the proof of Corollary 5.8. The penultimate equality is a result of $X_1,\ldots,X_m$ being an orthonormal basis. $\qquad\square$

6. Geometric insights from these expressions

The intrinsic curvatures in Section 5 are, by definition, independent of the model we choose and apply to $\operatorname{Gr}(k,\mathbb{R}^n)$ as an abstract manifold. Indeed, the results in this section will all be stated for the Grassmannian $\operatorname{Gr}(k,\mathbb{R}^n)$, as opposed to its involution model $\operatorname{Gr}(k,n)$. The expressions we found in Section 5 by way of the involution model $\operatorname{Gr}(k,n)$ permit us to concretely study the geometry of the abstract manifold $\operatorname{Gr}(k,\mathbb{R}^n)$, and thereby obtaining new geometric insights. Even without going out of our way to search for such insights, we can already see a few that, as far as we know, have never been observed before for the Grassmannian.

For example, we may deduce the following, which is mildly surprising because we do not even know how to define a Plebański tensor [39].

Corollary 6.1 (Plebański curvature). *The Plebański tensor of the Grassmannian is zero.*

An observant reader might have noticed that this is the only tensor mentioned in Section 1 whose definition did not appear in Section 3. The reason is that we do not know how to define the Plebański tensor in the coordinate-free manner adopted in modern mathematics. Every definition in the literature only gives its coordinates in terms of the coordinates of Z, the traceless Ricci curvature. But as we do know from Corollary 5.4 that $\mathsf{Z} = 0$ for the Grassmannian, its Plebański tensor must be zero as well.

An observant reader might also have noticed that several of the expressions in Table 1 are constant multiples of $\operatorname{tr}(XY)$. Therein lies two small results:

Corollary 6.2 (Codazzi tensors I)**.** *The Ricci, Schouten, and Bach curvatures of the Grassmannian are Codazzi.*

Proof. Corollaries 5.4, 5.6, and 5.9 show that the tensors in question are all constant multiples of g and therefore Codazzi since $\nabla \mathsf{g} = 0$. □

Corollary 6.3 (Codazzi tensors II)**.** *A symmetric bilinear form β on the Grassmannian with constant trace is Codazzi if and only if $\nabla\beta = 0$*

Proof. If $\nabla\beta = 0$, then it is Codazzi by definition. For the converse, we invoke the result [4] that any Codazzi tensor with constant trace on a compact Riemannian manifold must have $\nabla\beta = 0$ if the sectional curvature $\kappa \geq 0$ everywhere. By Corollary 5.3, the Grassmannian has $\kappa \geq 0$. □

The proof of Corollary 6.2 throws up another observation.

Corollary 6.4 (Divergence-free tensors)**.** *The Riemann and Weyl curvatures of the Grassmannian are divergence-free.*

Proof. By Corollary 5.4, Ric is a constant multiple of g and so $\nabla\mathsf{Ric} = 0$. Since the Riemann curvature is divergence-free if and only if the Ricci tensor is divergence-free and Codazzi [38, Corollary 9.4.5], it follows from Corollary 6.2 that $\operatorname{div}\mathsf{R} = 0$ for the Grassmannian. The relation [12, Equation 2-3] between Weyl and Cotton tensors

$$\operatorname{div}\mathsf{W} = \frac{\dim\mathcal{M} - 2}{\dim\mathcal{M} - 3}\mathsf{C}$$

for any manifold $\mathcal{M}$ of dimension at least four, taken together with Corollary 5.7 that $\mathsf{C} = 0$, yields $\operatorname{div}\mathsf{W} = 0$. For $\dim\mathcal{M} < 4$, W is identically zero [12, Remark 2.3]. □

The delta invariants obtained in Theorem 5.5 have never before been calculated for a manifold as complex as the Grassmannian. These values may look quotidian to the uninitiated, but they are not. We give an example to show how the value of $\underline{\delta}_{2,r}$ found in Theorem 5.5 vastly improves a classical result.

A *geodesic 2-sphere* is a 2-sphere S^2 embedded in a Riemannian manifold $\mathcal{M}$ as a totally geodesic submanifold. One fascinating fact about the geometry of the Grassmannian is that it contains a geodesic 2-sphere [29, 45, 46, 49]. This is a very unique property. For instance, $\mathbb{R}^3$ contains no geodesic 2-sphere, even though S^2 is, ironically, the unit sphere of $\mathbb{R}^3$. The reason is that S^2 is only a Riemannian submanifold but not a totally geodesic submanifold of $\mathbb{R}^3$.

A key result in [47] is that $\operatorname{Gr}(k, \mathbb{R}^n)$ contains *one* geodesic 2-sphere. We will show below that it in fact contains a product of *many* geodesic 2-spheres. To the best of our knowledge, this insight is new. It is also unusual. For instance, while the 3-sphere S^3 is known to contain a geodesic 2-sphere, it does not contain a product of more than one copy.

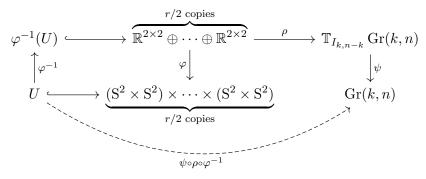**Theorem 6.5** (Embedding products of geodesic 2-spheres)**.** (a) *For any $r \leq \min\{\lfloor k/2 \rfloor, \lfloor n/4 \rfloor\}$, the product of r copies of S^2 can be embedded as a totally geodesic submanifold of $\operatorname{Gr}(k, n)$.*
(b) *For any $r \leq 2\lfloor k/2 \rfloor\lfloor (n-k)/2 \rfloor$, an open subset of the product of r copies of S^2 can be embedded as a totally geodesic submanifold of $\operatorname{Gr}(k, n)$.*

Proof. Let $r \leq \min\{\lfloor k/2 \rfloor, \lfloor n/4 \rfloor\}$ and $\mathbb{V}_1, \ldots, \mathbb{V}_r \subseteq \mathbb{R}^n$ be any r four-dimensional subspaces that are orthogonal to each other, i.e., $\mathbb{V}_j \subseteq \left(\bigoplus_{i \neq j} \mathbb{V}_i \right)^{\perp}$ for all $j = 1, \ldots, r$. Let $\mathbb{W}_0 \subseteq \left(\bigoplus_{i=1}^{r} \mathbb{V}_i \right)^{\perp}$ be a $(k - 2r)$-dimensional subspace. We define the embedding

$$\varepsilon : \mathrm{Gr}(2, \mathbb{V}_1) \times \cdots \times \mathrm{Gr}(2, \mathbb{V}_r) \to \mathrm{Gr}(k, \mathbb{R}^n), \quad (\mathbb{W}_1, \ldots, \mathbb{W}_r) \mapsto \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_r.$$

Clearly, the image of ε is totally geodesic. By [49, Section 4], since $\mathbb{V}_i$ is four-dimensional, $\mathrm{Gr}(2, \mathbb{V}_i)$ contains a geodesic 2-sphere[1] Σ_i^2 for each $i = 1, \ldots, r$. The restriction of ε to $\Sigma_1^2 \times \cdots \times \Sigma_r^2 \cong \mathrm{S}^2 \times \cdots \times \mathrm{S}^2$ (r copies) gives the desired embedding in (a).

For (b), we will need to use the involution model. By Corollary 5.3, $\kappa_Q(X, Y) \leq \frac{1}{4}$ for any $Q \in \mathrm{Gr}(k, n)$ and orthonormal $X, Y \in \mathbb{T}_Q \mathrm{Gr}(k, n)$. By [49, Theorem 5], $\kappa_Q(X, Y) = \frac{1}{4}$ if and only if X and Y are tangent vectors of a geodesic 2-sphere in $\mathrm{Gr}(k, n)$ passing through Q. Note that X and Y must span the tangent space of this geodesic 2-sphere. We set $k_1 := \lfloor k/2 \rfloor$, $k_2 := \lfloor (n-k)/2 \rfloor$ as in the proof of Theorem 5.5, and also $r := 2k_1 k_2$. Consider the commutative diagram

where φ, ψ are the exponential maps on the respective tangent spaces, U is an open subset of $\mathrm{S}^2 \times \cdots \times \mathrm{S}^2$ (r copies) on which φ^{-1} is well-defined, and ρ is the linear map defined by

$$(B_{ij})_{i,j=1}^{k_1, k_2} \mapsto \begin{bmatrix} 0 & \cdots & 0 & B_{1,1} & \cdots & B_{1, k_2+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{k_1+1, 1} & \cdots & B_{k_1+1, k_2+1} \\ B_{1,1}^{\mathsf{T}} & \cdots & B_{k_1+1, 1}^{\mathsf{T}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{1, k_2+1}^{\mathsf{T}} & \cdots & B_{k_1+1, k_2+1}^{\mathsf{T}} & 0 & \cdots & 0 \end{bmatrix}$$

where we have used the same notation as in the proof of Theorem 5.5 and set B_{pq} to be the zero matrix if either $p = k_1 + 1$ or $q = k_2 + 1$. As in the proof of Theorem 5.5, we may choose an orthonormal basis X_1, Y_1, X_2, Y_2 for each copy of $\mathbb{R}^{2 \times 2}$ such that $\kappa_Q(\rho(X_1), \rho(Y_1)) = \kappa_Q(\rho(X_2), \rho(Y_2)) = \frac{1}{4}$. By shrinking U if necessary, we may assume that ψ is injective on $\rho(\varphi^{-1}(U))$. Hence $\psi \circ \rho \circ \varphi^{-1}(U)$ is the required open subset in (b). $\square$

Suppose k and n are both even. Then the upper bound in Theorem 6.5(b) is $r = k(n-k)/2$. This is sharp as the dimension of the product of $r + 1$ copies of S^2 is $k(n-k) + 2$ and it exceeds the dimension of $\mathrm{Gr}(k, n)$; so a product of $r + 1$ copies of S^2 cannot be embedded in $\mathrm{Gr}(k, n)$.

7. WHY WE FAVOR THE INVOLUTION MODEL

As we alluded to in Section 1, a secondary goal of this article is to demonstrate the advantages of using the involution model (2). Here we will make some comparisons with other common models of the Grassmannian in algebraic geometry (Section 7.1), differential geometry (Section 7.2), and integral geometry (Section 7.3).

---

[1]When $\mathbb{V}$ is four-dimensional, any maximal subset of mutually isoclinic 2-planes in $\mathrm{Gr}(2, \mathbb{V})$ is a geodesic 2-sphere.

To elaborate, as an abstract manifold, the Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$ is just the set of k-planes in $\mathbb{R}^n$. While any manifold, by definition, can be given local coordinates, experience tells us that they are rarely useful beyond basic proofs — nobody really works with charts and atlases outside a first course in differential geometry. Especially in applied mathematics, but also in pure mathematics, the preferred approach is to give $\mathrm{Gr}(k, \mathbb{R}^n)$ a system of global, extrinsic coordinates that are easier to work with — this is what we mean by a *model* for $\mathrm{Gr}(k, \mathbb{R}^n)$.

7.1. **Plücker model.** The standard model of the Grassmannian in algebraic geometry (see [21, Lecture 6] and [41, Chapter 1, Section 4.1]) is as the set of rank-one alternating tensors in projective space, i.e., the image of the Plücker embedding:

$$\mathrm{Gr}(k, \mathbb{R}^n) \cong \big\{ [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\Lambda^k(\mathbb{R}^n)) : v_1, \ldots, v_k \in \mathbb{R}^n \text{ linearly independent} \big\}.$$

While this has some desirable mathematical properties [31, Section 1], its main issue is that the ambient space $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$ is a manifold of exceedingly high dimension $\binom{n}{k} - 1$. This not only presents a computational conundrum but also results in complex expressions for even relatively basic quantities. For example, the second fundamental form has been derived in [1, Lemma 2.1 and Proposition 2.3] for the Plücker model and both its calculation and the expression are significantly more involved than those appearing in this article. In fact for even moderate values of k the expressions in [1] are next-to-impossible to use or even compute since they involve lengthy sums of high order tensors.

One observation from the extrinsic curvatures calculated in Section 4 is that the involution model is extremely unlike the Plücker model. For example, the mean curvature of the image of the Plücker embedding is well-known to be zero but it is far from zero in the involution model, as we saw in Corollary 4.6. Given that the mean curvature is determined by the second fundamental form, this shows that the second fundamental forms of both models must be different and therefore so are their Gaussian and principal curvatures.

7.2. **Quotient models.** The most common models of the Grassmannian in differential geometry (see [25, Chapter VII] and [8, Chapter 9]) are as one of several quotient spaces:

$$
\begin{aligned}
\mathrm{Gr}(k, \mathbb{R}^n) &\cong \mathrm{O}(n) / \big(\mathrm{O}(k) \times \mathrm{O}(n-k) \big) \\
&\cong \mathrm{V}(k, n) / \mathrm{O}(k) \\
&\cong \mathrm{GL}(n) / \big(\mathrm{GL}(k) \times \mathrm{GL}(n-k) \big) \\
&\cong \mathrm{St}(k, n) / \mathrm{GL}(k),
\end{aligned}
$$

(16)

where $\mathrm{V}(k, n) := \{ V \in \mathbb{R}^{n \times k} : V^{\mathsf{T}} V = I \}$ and $\mathrm{St}(k, n) := \{ X \in \mathbb{R}^{n \times k} : \mathrm{rank}(X) = k \}$ are two common models for the Stiefel manifold of k-frames in $\mathbb{R}^n$. As usual $\mathrm{O}(n) = \mathrm{V}(n, n)$ and $\mathrm{GL}(n) = \mathrm{St}(n, n)$ denote the orthogonal and general linear groups respectively.

By exploiting their homogeneous space structures, the more basic intrinsic curvatures such as Riemann, Ricci, and sectional curvatures of $\mathrm{Gr}(k, \mathbb{R}^n)$ are standard calculations that are classical in differential geometry [9, 16, 40]. However, the use of quotient spaces inevitably gives rise to formulas involving horizontal lifts of tangent vectors and arbitrary representatives of equivalence classes. This introduces layer upon layer of ambiguities requiring multiple arbitrary choices.

We will walk the reader through the calculation of Riemann curvature in $\mathrm{O}(n) / \big(\mathrm{O}(k) \times \mathrm{O}(n-k) \big)$ to illustrate the case in point. We will delimit equivalence classes in $\llbracket \cdot \rrbracket$ below. In this model,

(i) a point $\llbracket Q \rrbracket \in \mathrm{O}(n) / \big(\mathrm{O}(k) \times \mathrm{O}(n-k) \big)$ is a coset

$$\llbracket Q \rrbracket = \left\{ Q \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in \mathrm{O}(n) : (Q_1, Q_2) \in \mathrm{O}(k) \times \mathrm{O}(n-k) \right\},$$

for some $Q \in \mathrm{O}(n)$ but Q is not canonically given;

(ii) a tangent vector $[\![X]\!]_{[\![Q]\!]} \in \mathbb{T}_{[\![Q]\!]} \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$ is an equivalence class of pairs

$$[\![X]\!]_{[\![Q]\!]} = \Big\{\big(Q, X + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\big) \in \operatorname{O}(n) \times \mathfrak{so}(n)/\big(\mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\big)\Big\}\Big/ \sim$$

where the equivalence relation is defined by

$$\big(Q, X + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\big) \sim \big(Q', X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\big)$$

if and only if there is some $(Q_1, Q_2) \in \operatorname{O}(k) \times \operatorname{O}(n-k)$ with

$$Q' = Q \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

(17)

$$X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k) = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}^{\mathsf{T}} \big(X + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\big) \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

Evidently, in this model even an object as basic as a tangent vector is an equivalence class (defined by $\sim$) of equivalence classes (the coset $X + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$). Every layer of equivalence relations introduces a layer of ambiguity but more importantly it often takes additional effort in the form extra calculations or computations.

Writing down a tangent vector $[\![X]\!]_{[\![Q]\!]}$ as a pair of actual matrices (Q_0, X_0) requires making three arbitrary choices: first a representative Q_0 of $[\![Q]\!]$, followed by a representative $(Q', X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k))$ of $[\![X]\!]_{[\![Q]\!]}$, and finally a representative X_0 of $X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$. Note that these cannot be chosen arbitrarily nor a priori but need to satisfy (17). We will give the details below.

We begin by picking a representative $Q_0 \in \pi^{-1}([\![Q]\!]) \subseteq \operatorname{O}(n)$ where $\pi : \operatorname{O}(n) \to \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$ is the quotient map, a Riemannian submersion. To construct the horizontal lift $X_0 \in \mathbb{T}_Q \operatorname{O}(n)$ of a tangent vector $[\![X]\!]_{[\![Q]\!]} \in \mathbb{T}_{[\![Q]\!]} \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$, the recommendation in the classic article of Edelman–Arias–Smith [17] is to use the isomorphism

$$d_Q\pi : \left\{ Y \in \mathbb{R}^{n\times n} : Q^{\mathsf{T}} Y = \begin{bmatrix} 0 & B \\ -B^{\mathsf{T}} & 0 \end{bmatrix}, \; B \in \mathbb{R}^{k\times(n-k)} \right\} \to \mathbb{T}_{[\![Q]\!]} \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big),$$

defined for any $Q \in \operatorname{O}(n)$ and $[\![X]\!]_{[\![Q]\!]} \in \mathbb{T}_{[\![Q]\!]} \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$, and then compute the horizontal lift as $X_0 := (d_{Q_0}\pi)^{-1}([\![X]\!]_{Q_0})$.

To get X_0 explicitly as a matrix, we will need to pick a representative $(Q', X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k))$ for $[\![X]\!]_{[\![Q_0]\!]}$ followed by a representative $\widehat{X}$ of $X' + \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$). Observe that we cannot simply set Q' to be Q_0 since $(Q_0, \widehat{X})$ will not satisfy (17) in general. Indeed, as we require

$$(Q', \widehat{X}) \sim (Q_0, (Q'^{\mathsf{T}} Q_0)^{\mathsf{T}} \widehat{X}(Q'^{\mathsf{T}} Q_0)),$$

we will need to compute X_0 as

$$X_0 = Q_0(Q'^{\mathsf{T}} Q_0)^{\mathsf{T}} \begin{bmatrix} 0 & \widehat{B} \\ -\widehat{B}^{\mathsf{T}} & 0 \end{bmatrix} (Q'^{\mathsf{T}} Q_0) = Q' \begin{bmatrix} 0 & \widehat{B} \\ -\widehat{B}^{\mathsf{T}} & 0 \end{bmatrix} Q'^{\mathsf{T}} Q_0,$$

with $\widehat{B}$ the upper right $k \times (n-k)$ submatrix of $\widehat{X}$.

It might appear that to compute the Riemann curvature[2] at $[\![Q]\!] \in \operatorname{O}(n) / \big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$, we simply follow the procedure above to compute horizontal lifts $X, Y, Z, W \in \mathbb{T}_Q \operatorname{O}(n)$ of $[\![X]\!]_{[\![Q]\!]}, [\![Y]\!]_{[\![Q]\!]}, [\![Z]\!]_{[\![Q]\!]}, [\![W]\!]_{[\![Q]\!]} \in \mathbb{T}_{[\![Q]\!]} \operatorname{O}(n)/\big(\operatorname{O}(k) \times \operatorname{O}(n-k)\big)$ and evaluate the expression on the right hand-side:

(18)
$$\operatorname{Rie}([\![X]\!], [\![Y]\!], [\![Z]\!], [\![W]\!]) = -\frac{1}{4}\langle [[X,Y], Z], W \rangle.$$

---

[2]The expression comes from applying the standard method for calculating Riemann curvature on a quotient model of any symmetric space [23, Theorem 4.2].

But this is a notational illusion. Even if we start from the same representative $Q_0 \in \pi^{-1}(\llbracket Q \rrbracket) \subseteq$ $\mathrm{O}(n)$, the procedure above will yield horizontal lifts of tangents vectors at *different* representatives $Q'_X, Q'_Y, Q'_Z, Q'_W \in \pi^{-1}(\llbracket Q \rrbracket)$. There is no guarantee that $Q'_X = Q'_Y = Q'_Z = Q'_W = Q_0$ and extra steps are necessary to align these different representatives. To align Q'_X with Q_0, we need to compute the horizontal lift of $\llbracket X \rrbracket_{\llbracket Q \rrbracket}$ at Q_0 as

$$(d_{Q_0}\pi)^{-1}(\llbracket X \rrbracket_{\llbracket Q \rrbracket}) = Q_0 P_X^\mathsf{T} Q_X'^\mathsf{T} (d_{Q'_X}\pi)^{-1}(\llbracket X \rrbracket_{\llbracket Q \rrbracket}) P_X = (d_{Q'_X}\pi)^{-1}(\llbracket X \rrbracket_{\llbracket Q \rrbracket}) P_X,$$

where $P_X := (Q'_X)^\mathsf{T} Q_0$. This process has to be repeated four times to get

$$(d_{Q_0}\pi)^{-1}(\llbracket X \rrbracket_{\llbracket Q \rrbracket}), \quad (d_{Q_0}\pi)^{-1}(\llbracket Y \rrbracket_{\llbracket Q \rrbracket}), \quad (d_{Q_0}\pi)^{-1}(\llbracket Z \rrbracket_{\llbracket Q \rrbracket}), \quad (d_{Q_0}\pi)^{-1}(\llbracket W \rrbracket_{\llbracket Q \rrbracket})$$

before we may evaluate (18).

Contrast this with Proposition 5.1, where the calculation of Riemann curvature in the involution model avoids all these issues and the derivation of its expression is essentially a one-liner. It is important to point out that although the expression in (18) superficially resembles our expression in Proposition 5.1, this is also a notational illusion — they are completely different. The easiest way to see this is by observing that the matrices in (18) are all skew-symmetric whereas those in Proposition 5.1 are all symmetric.

The goal of Edelman, Arias, and Smith in [17] is to extend line search optimization methods to a function $f : \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k)) \to \mathbb{R}$. By and large this permits them to work with one search direction, i.e., a single tangent vector $\llbracket X \rrbracket_{\llbracket Q \rrbracket}$, at every point $\llbracket Q \rrbracket$. As a result, one could get around the problem by optimizing $f \circ \pi : \mathrm{O}(n) \to \mathbb{R}$ along horizontal directions. In the calculation of curvatures, we are required to work with four tangent vectors simultaneously and thus the alignment of different representatives of a given pair of equivalence classes cannot be avoided.

Although we have elected to make our point with $\mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$, the issues identified above apply to every quotient model in (16). The root of these issues is that there is no global way to describe tangent vectors of $\mathrm{Gr}(k, \mathbb{R}^n)$ in any of these quotient models. Indeed, the absence of such a global description is the reason why expressions for various curvatures in [48, 49, 9, 40, 16] can only be given locally. The same reason accounts for the expediency of the involution model — not only does it describe all points on the Grassmannian (2), it describes all tangent vectors at all point in a single unified way (9).

7.3. Projection model.
The standard model of the Grassmannian in integral geometry (see [36, Chapter 9] and [34, Chapter 3]) is as the set of projection matrices:

$$(19) \qquad\qquad \mathrm{Gr}(k, \mathbb{R}^n) \cong \{P \in \mathbb{S}^n : P^2 = P, \ \mathrm{tr}(P) = k\}.$$

This is the model closest to the involution model. Indeed we showed in [31, Theorem 6.1] that they are two instances in an infinite family of such models parameterized by the condition number of matrices used. However they are also on two opposite ends: The projection model is the unique model in this family that represents points as singular matrices (infinitely ill-conditioned) whereas the involution model is the unique model in this family that represents points as orthogonal matrices (perfectly conditioned). Every other model in this family represents points with matrices of condition number strictly between one and infinity.

In numerical computations, methods based on projection matrices [44, Lecture 8] are well-known to be significantly less stable than methods based on orthogonal matrices [44, Lecture 10] — in fact this comparison is famously used to illustrate numerical stability of algorithms.

In hand calculations, the singularity of projection matrices in (19) is a handicap, especially when contrasted against the ease of inverting orthogonal matrices (with a reminder that matrices in (2) are automatically orthogonal).

The equations defining (19) are also less convenient than those defining (2). Any calculations involving tangent vectors in the involution model would require one to differentiate $Q^2 = I$ to get

$XQ + QX = 0$. But doing the same in the projection model would require one to differentiate $P^2 = P$ to get $XP + PX = X$. The latter is more difficult to use than the former; indeed any calculation involving the latter would usually involve a change of coordinates $P \mapsto 2P - I$ to simplify but that yields exactly the involution model since $Q = 2P - I$ [27, Proposition 3.5]. Nevertheless, because of this relation between the two models, every expression we derived for the involution model in this article gives one for the projection model, up to a constant factor.

The result [19, Proposition 13] comes close to obtaining the principal curvatures in Corollary 4.7 using the projection model. Nevertheless, while (19) was used to model the manifold, the tangent spaces in [19] were still modeled as horizontal spaces in the quotient model $\mathrm{O}(n)/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big)$, making the messy calculations in Section 7.2 all but unavoidable.

8. Conclusion

In studying curvatures, it is helpful to have an illuminating instance of a manifold $\mathcal{M}$ where all different forms of curvatures can be explicitly calculated and compared side-by-side like in Table 1. We are unaware of any nontrivial examples of this in the literature. The reason is clear in retrospect: For many of these curvatures, their defining equations in terms of local coordinates are near-impossible to calculate for anything more complex than a sphere. But even when embedded in $\mathbb{R}^m$, which one may do with Nash embedding, the resulting extrinsic coordinates are still difficult to use. The key to our simple formulas in this article is that we have embedded our manifold in a space of matrices, and matrices are endowed with far richer structures — we may multiply or decompose them; impose orthogonality or symmetry on them; calculate their determinant, norm, or rank; find their eigen- or singular values and vectors; among a myriad of yet other features

Future work. The multitude of curvatures discussed in this article might lead the reader to think that we have exhausted the topic. This is not the case.

Some tensors are beyond our reach. The *obstruction tensor* [18, Equation 3.25] is a 2-tensor that equals the Bach tensor in Corollary 5.9 for four-dimensional manifolds but $\dim \mathrm{Gr}(k,n) \neq 4$ if $(k,n) \notin \{(1,5),(2,4)\}$. The *Lanczos tensor* [28] is a 3-tensor that is an antiderivative of the Weyl tensor in Corollary 5.8 and defined as a solution to a partial differential equation, which we are not even sure has a solution for $\mathrm{Gr}(k,n)$. The *Bel tensor* [7] is a 4-tensor constructed by contracting the Riemann tensor with itself in all possible ways and the *Bel–Robinson tensor* does the same with the Weyl tensor but we are unable to simplify their expressions for $\mathrm{Gr}(k,n)$ to the extent of those in Table 1.

Although we have limited the discussions in this article to the Levi-Civita connection, the natural choice from the perspective of Riemannian geometry, we saw in Section 3.3 that there are other alternatives if we study the Grassmannian in the context of non-Riemannian geometry. In this case the torsion, nonmetricity, and cocurvature tensors discussed at the end of Section 3.2 may no longer be zero. Using a different connection permits us to study yet other curvatures like the *contorsion tensor* [6, Theorem 6.2.5], a 3-tensor that quantifies its deviation from Levi-Civita.

When presented with a complicated d-tensor $T \in \mathbb{V}^{\otimes d}$, a common gambit in mathematics and physics [2, 5, 33, 42, 43] is to decompose it by decomposing the space in which it lies. More precisely, for any group $G \subseteq \mathrm{GL}(\mathbb{V})$, we decompose $\mathbb{V}^{\otimes d} = \bigoplus_{\lambda \in \widehat{G}} \mathbb{V}_\lambda$ into irreducible G-submodules, giving a decomposition $T = \sum_{\lambda \in \widehat{G}} T_\lambda$ with $T_\lambda \in \mathbb{V}_\lambda$. An example is the *Ricci decomposition* of the Riemann curvature into the scalar, traceless Ricci, and Weyl curvatures [5, Chapter 1, Section G],

$$\mathsf{Rie} = \mathsf{W} + \frac{1}{k(n-k)-2}\mathsf{Z} \oslash \mathsf{g} + \frac{\mathsf{Sca}}{2k(n-k)\big(k(n-k)-1\big)}\mathsf{g} \oslash \mathsf{g},$$

which we have implicitly used in our definition of W; here $d = 4$, $\mathbb{V} = \mathbb{T}_Q \mathrm{Gr}(k,n)$, and $G = \mathrm{O}\big(k(n-k)\big)$. It would be interesting to find similar relations among the curvatures in Table 1.

Last but not least, while manifold optimization is not one of our goals here, it remains at the back of our minds. Existing optimization algorithms almost exclusively rely on two quantities — gradient and Hessian. As shown in [13], it is certainly conceivable to use, say, the second fundamental form to optimize a smooth function. This and other curvatures computed in Table 1 may turn out to be useful in this regard.

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