# A Combinatorial, Primal-Dual Approach to Semidefinite Programs

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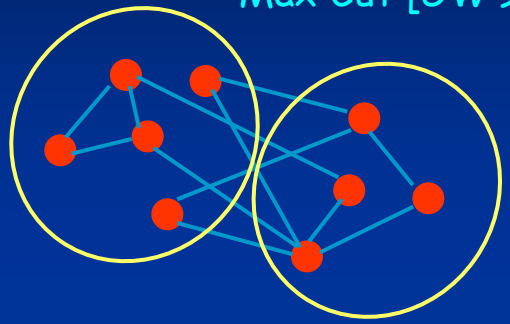
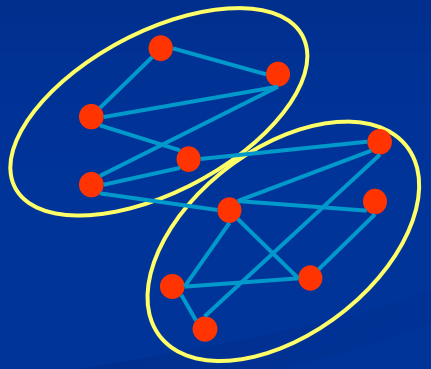
# The Ubiquity of Semidefinite Programming

Max Cut [GW'95]

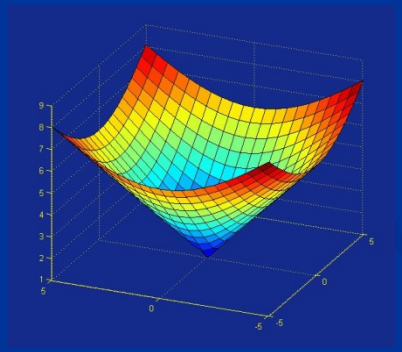


Balanced Partitioning [ARV'04]



$dx(t)/dt = Ax(t)$

Control Theory



SDP

$(a \vee \neg b) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg c \vee b)$

1 0 0 0 1 1 0 1

Constraint Satisfaction [ACMM'05]

Graph Coloring [KMS'98, ACC'06]

# Algorithms for SDP

Ellipsoid

- $O(n^8)$ it

...ethods

...time

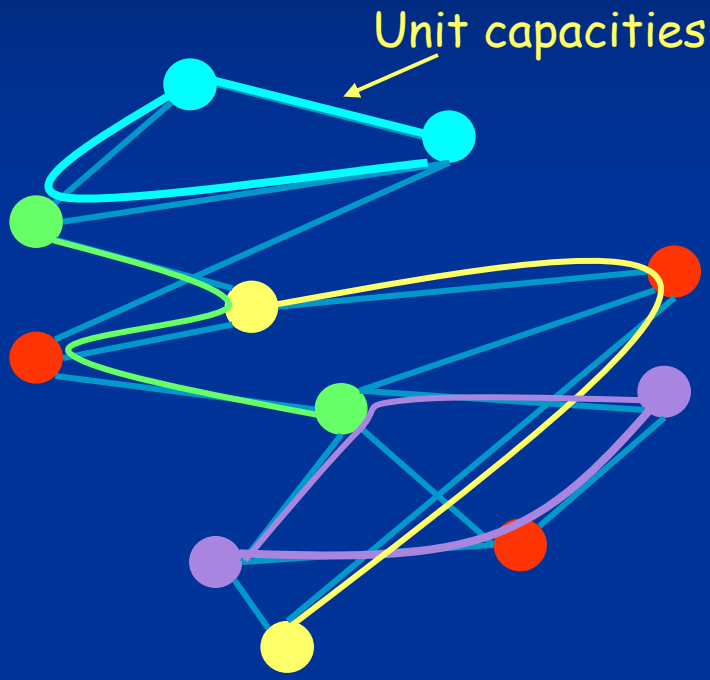
...]:

**MISSING**

Combinatorial,
Primal-Dual algorithms

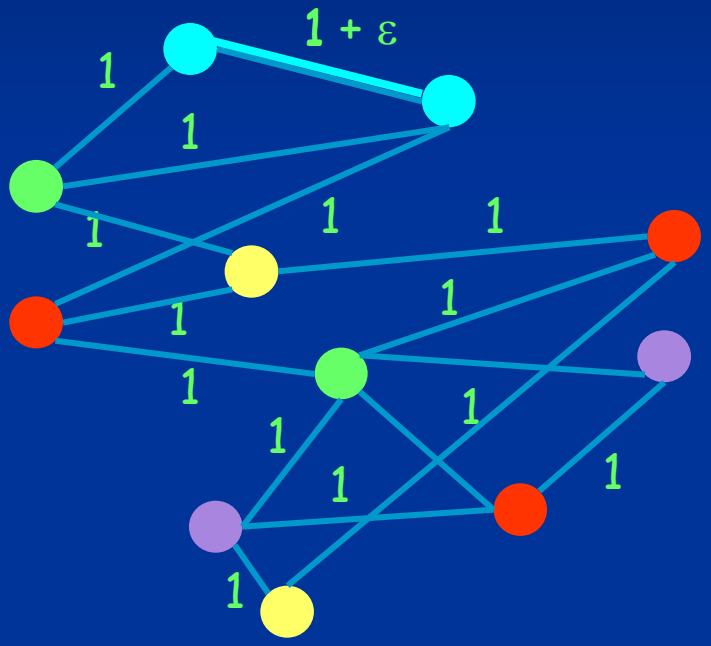
- Reduction to eigenvectors
- poly($1/\varepsilon$) dependence on $\varepsilon$, limits applicability (e.g. Sparsest Cut)

# Primal-Dual algorithms for LP: Multicommodity Flows




Objective: Maximize total flow, while respecting edge capacities

# Primal-Dual algorithms for LP: Multicommodity Flows




- Edge weights $w_e = 1$
- Repeat until some $e$ reaches capacity:
  - Find shortest path $p$
  - Route $\varepsilon^2/\log(m)$ flow on $p$
  - Update $w_e$ for all $e \in p$ as

$$w_e \leftarrow w_e \cdot (1 + \varepsilon)$$

- Output flow.

Thm [GK'98]: Stops in $\tilde{O}(m)$ rounds with $(1 - 2\varepsilon)\cdot OPT$ flow. Total running time is $\tilde{O}(m^2)$.

# Primal-Dual algorithms for LP: Multicommodity Flows

1. Primal-Dual algorithm
2. Combinatorial
3. Multiplicative Weights Update Rule

- Edge weights $w_e = 1$
- Repeat until some $e$ reaches capacity:
  - Find shortest path $p$
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Thm [GK'98]: Stops in $\tilde{O}(m)$ rounds with $(1 - 2\varepsilon)\cdot OPT$ flow. Total running time is $\tilde{O}(m^2)$.

- Starting point for our work:

  Analogous primal-dual algorithms for SDP?

- Difficulties:
  - Positive semidefiniteness hard to maintain
  - Rounding algorithms exploit geometric structure (e.g. negative-type metrics)
  - Matrix operations inefficient to implement

- Our work: a generic scheme that yields fast, combinatorial, primal-dual algorithms for various optimization problems using SDP

# Our Results: Primal-Dual algorithms

| Problem | Previous best: $O(\sqrt{\log n})$ apx | Our alg.: $O(\sqrt{\log n})$ apx | Our alg.: $O(\log n)$ apx |
|---|---|---|---|
| Undirected Sparsest Cut | $\tilde{O}(n^2)$ [AHK'04] | $\tilde{O}(n^2)$ | $\tilde{O}(m + n^{1.5})$ |
| Undirected Balanced Sep. | $\tilde{O}(n^2)$ [AHK'04] | $\tilde{O}(n^2)$ | $\tilde{O}(m + n^{1.5})$ |
| Directed Sparsest Cut | $\tilde{O}(n^{4.5})$ | $\tilde{O}(m^{1.5} + n^{2+\varepsilon})$ | $\tilde{O}(m^{1.5})$ |
| Directed Balanced Sep. | $\tilde{O}(n^{4.5})$ | $\tilde{O}(m^{1.5} + n^{2+\varepsilon})$ | $\tilde{O}(m^{1.5})$ |
| Min UnCut | $\tilde{O}(n^{4.5})$ | $\tilde{O}(n^3)$ | --- |
| Min 2CNF Deletion | $\tilde{O}(n^{4.5})$ | $\tilde{O}(nm^{1.5} + n^3)$ | --- |

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Cannot be achieved by LPs even!

# Our Results: Near-linear time algorithm for Max Cut

- $\tilde{O}(n + m)$ time algorithm to approximate Max Cut SDP to $(1 + o(1))$ factor

- Previous best: $\tilde{O}(n^2)$ time algorithm by Klein, Lu '95

# Prototype MW for LPs: Winnow [L'88]

$a_1 \cdot x \geq -\delta$
$a_2 \cdot x \geq -\delta$
$\vdots \quad \vdots$
$a_m \cdot x \geq -\delta$
$x \geq 0$
$\sum_i x_i = 1$

Thm: Finds x in $O(\rho^2 \log(n)/\delta^2)$ iterations.

$\rho = \max_{ij} |a_{ij}|$

MW for LP:
Init: $x = (1/n, \ldots, 1/n)$
Update:
$x_j = x_j \cdot \exp(-\varepsilon\, a_{ij})/\Phi$
$\Phi$ = normalization factor

$x$

$a_i$

Oracle
Find i s.t. $a_i \cdot x < -\delta$

MW algorithm: boosting, hard-core sets, zero-sum games, flows, portfolio, …

Convex comb. of constraints allowed:
$\sum_i y_i\, a_i \cdot x < -\delta$, where $y_i \geq 0$, $\sum_i y_i = 1$.
Hence Primal-Dual.

# Prototype Matrix MW for SDPs

$$A_1 \bullet X \geq -\delta$$
$$A_2 \bullet X \geq -\delta$$
$$\vdots \qquad \vdots$$
$$A_m \bullet X \geq -\delta$$
$$X \succeq 0$$
$$\mathrm{Tr}(X) = 1$$

Thm: Finds X in $O(\rho^2 \log(n)/\delta^2)$ iterations.

$\rho = \max_i \|A_i\|$

Matrix MW for SDP:
Init: $X = I/n$
Update:
$X = \exp(-\varepsilon \sum_{s=1}^{t} A_{i_s})/\Phi$
$\Phi$ = norm. factor

Analysis uses $\Phi$ as potential fn.

$X$

$A_{i_t}$

Oracle
Find $i_t$ s.t. $A_{i_t} \bullet X < -\delta$

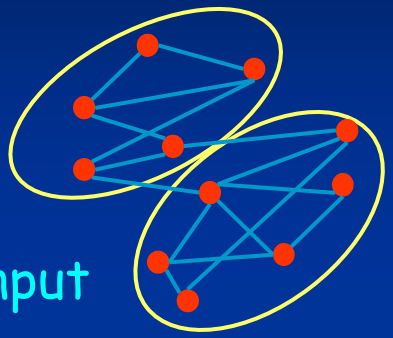
Convex comb. of constraints allowed:
$\sum_i y_i A_i \bullet X < -\delta$, where $y_i \geq 0$, $\sum_i y_i = 1$.
Hence Primal-Dual.

# The Matrix Exponential

- Matrix exponential:
  $\exp(A) = I + A + A^2/2! + A^3/3! + \ldots$
- Always PSD: $\exp(A) \succeq 0$
- Golden-Thompson inequality:
  $\mathrm{Tr}(\exp(A+B)) \leq \mathrm{Tr}(\exp(A)\exp(B))$
- Computation:
  - $O(n^3)$ time
  - Usually, can use J-L lemma
  - Only need $\exp(A)v$ products: $\tilde{O}(m)$ time

Matrix MW for SDP:
Init: $X = I/n$
Update:
$X = \exp(-\varepsilon \sum_{s=1}^{t} A_{i_s})/\Phi$
$\Phi$ = norm. factor

# Approximation via SDP Relaxations



Input

Reduction

0-1 Quadratic Program

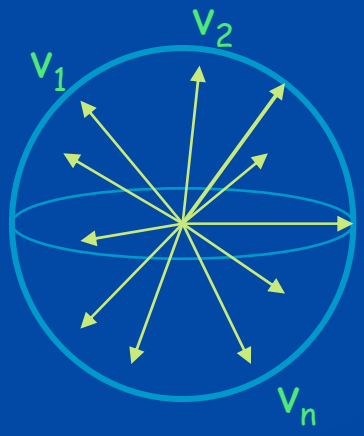
Relaxation

SDP: Vector vars

SDP Solver

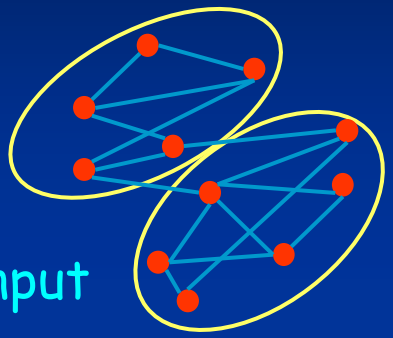
Primal-Dual SDP

Rounding Algorithm




SDP Opt

# Primal-Dual SDP Framework



Input

Reduction

0-1 Quadratic Program

Relaxation

SDP: Vector vars

Primal-Dual SDP

Rounding Algorithm

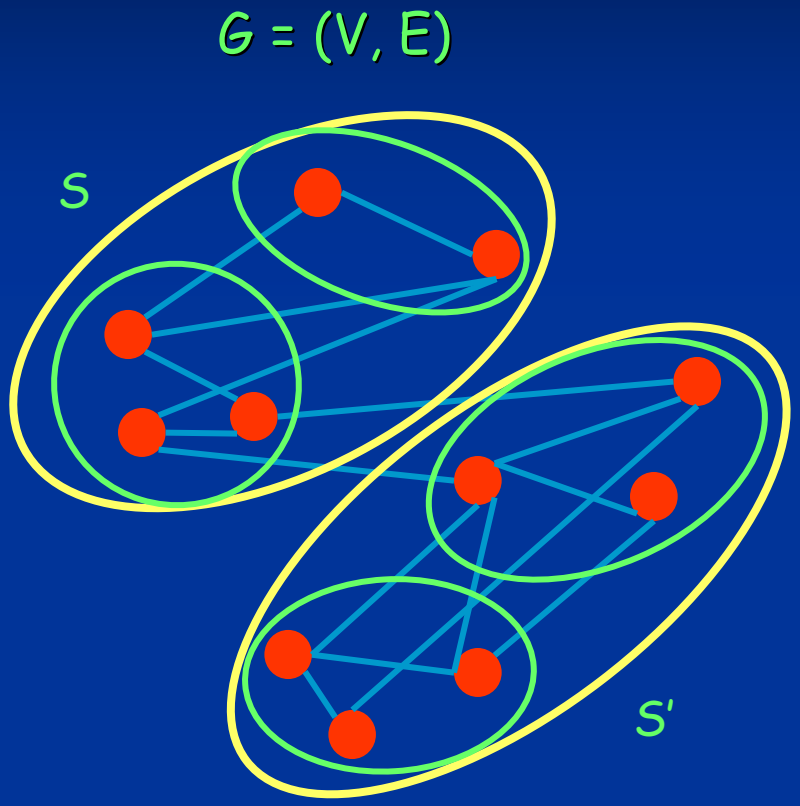
$v_1$ $v_2$ $v_n$

$Y_1, \ldots, Y_m$

$X$

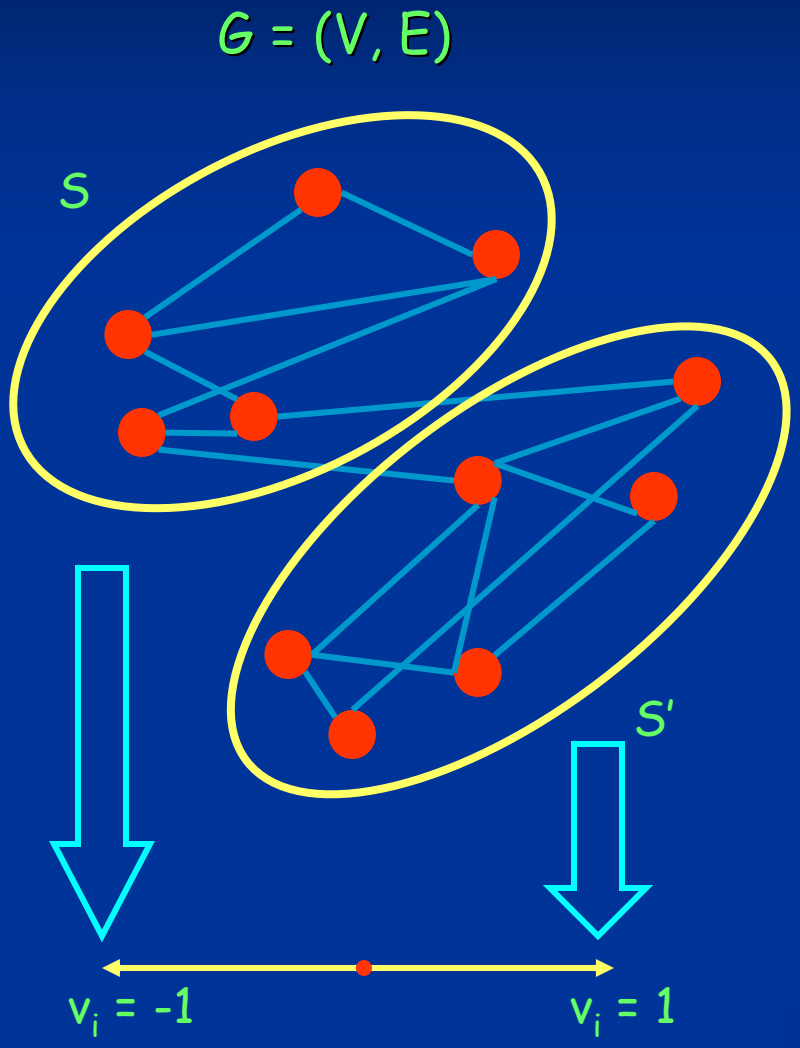
Matrix MW

# Approximating Balanced Separator




- Cut $(S, S')$ is c-balanced if $|S|, |S'| \geq cn$
- Min c-Balanced Separator: c-balanced cut of min capacity
- Numerous applications:
  - Divide-and-conquer algorithms
  - Markov chains
  - Geometric embeddings
  - Clustering
  - Layout problems
  - …

# SDP for Balanced Separator




$\frac{1}{4}\|v_i - v_j\|^2 = 0$ if i, j on same side,

$= 1$ otherwise

SDP:

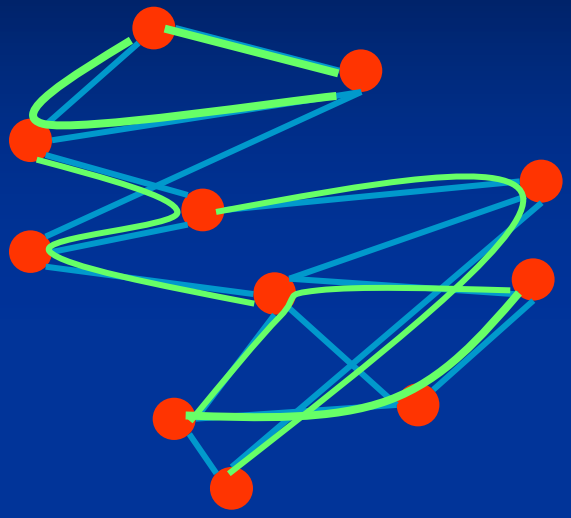
$$\min \sum_{i,j \in E} \tfrac{1}{4}\|v_i - v_j\|^2$$

$$\forall i: \ \|v_i\|^2 = 1$$

$$\forall ijk: \ \|v_i - v_j\|^2 + \|v_j - v_k\|^2 \geq \|v_i - v_k\|^2$$

$$\sum_{i,j} \tfrac{1}{4}\|v_i - v_j\|^2 \geq c(1-c)n^2$$

# Implementing Oracle



Primal:

$\min \sum_{i,j \in E} \frac{1}{4}\|v_i - v_j\|^2$

$\forall i: \ \|v_i\|^2 = 1$

$\forall ijk: \ \|v_i - v_j\|^2 + \|v_j - v_k\|^2 \geq \|v_i - v_k\|^2$

$\sum_{i,j} \frac{1}{4}\|v_i - v_j\|^2 \geq c(1-c)n^2$

Thm: Given $v_i$, $\alpha$:

1. Max-flow $\Rightarrow$ desired flow or cut of value $O(\log(n)\cdot\alpha)$.
2. Multicommodity flow $\Rightarrow$ desired flow or cut of value $O(\sqrt{\log(n)}\cdot\alpha)$.

# Conclusions and Future Work

- Matrix MW algorithm: more applications in
  - Solving SDPs: e.g. Min Linear Arrangement
  - Quantum algorithms: density matrix is a central concept
  - Learning: e.g. online variance minimization [WK COLT'06], online PCA [WK NIPS'06]
  - Other applications?

- Linear time algorithm for Sparsest Cut?
  - Our algorithm runs in $\tilde{O}(m + n^{1.5})$ time for an $O(\log n)$ approximation

Thank you!