Rank According to Perron: A New Insight

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If $A=(a_{ij})$, $a_{ij}>0$, $i, j=1,\dots,n$, Perron [4] proved that A has a real positive eigenvalue $\lambda_{\max}$ (called the principal eigenvalue of A) that is unique, and $\lambda_{\max}>|\lambda_k|$ for the remaining eigenvalues of A. Furthermore, the principal eigenvector $w=(w_1,\dots,w_n)$ that is a solution of $Aw=\lambda_{\max}w$ is unique to within a multiplicative constant and $w_i>0$, $i=1,\dots,n$. We can make the solution w unique through normalization. We define the norm of the vector w as $\|w\|=we$ where $e=(1,1,\dots,1)^T$, e^T is its transpose, and to normalize w is to divide it by its norm. We shall always think of w in normalized form.

Perron's result has found wide use in many areas, both theoretical and applied. Among these are applications to multicriterion decisions. In a recent article in this journal [1], Barbeau gave a lucid exposition and illustrated the use of the Analytic Hierarchy Process (AHP) to assist a college in interpreting grades and testing students for admission. The AHP, which I developed, uses Perron's principal eigenvector as an essential property for determining the preference ranking among a set of alternatives when the judgments are inconsistent. In the AHP the alternatives are compared in pairs with respect to a common attribute in a positive reciprocal matrix whose entries represent the numerical value of the relative preference for one (the row) over another (the column). The reader is referred to Barbeau's paper for a concise reference to the AHP. My purpose in this note is to show how important the principal eigenvector is in determining the rank of the alternatives through dominance walks.

Suppose we wish to rate five teachers A, B, C, D, and E according to their excellence in teaching. We enter our evaluation in the following matrix, whose principal eigenvector has been normalized:

	A	B	C	D	E	Eigenvector solution
A	1	1/6	1/2	1/9	5	.0893
B	6	1	2	1	5	.3287
C	2	1/2	1	1	5	.1983
D	9	1	1	1	5	.3413
E	1/5	1/5	1/5	1/5	1	.0424

This matrix gives the pairwise dominance of the alternative in the row over that in the column. For example, teacher B is rated to be 2 times better than teacher C, which is the entry in the $(2,3)$ position. The reciprocal value of $1/2$ is then entered in the transpose position which here is $(3,2)$. From this pairwise comparison matrix we wish to derive a scale of relative standing for the teachers. At first one may think that this is given by adding the components of each row and normalizing the result. This is only true if the matrix is consistent ($a_{ij}a_{jk}=a_{ik}$, $i, j, k=1,\dots,n$) for then the matrix has unit rank and any row is a multiple of a single row. The above matrix is inconsistent. For example, $a_{23}=2\neq a_{13}/a_{12}=(1/2)/(1/6)=3$. Note that consistency implies the reciprocal relation but not conversely. Let us now examine the general case.

There is a natural way to derive the rank order of a set of alternatives from a pairwise comparison matrix A [**6**]. The rank order of each alternative is the relative proportion of its dominance over the other alternatives. This is obtained by adding the elements in each row in A and dividing by the total over all the rows. However, A only captures the dominance of one alternative over each other in one step. But an alternative can dominate a second by first dominating a third alternative, and then the third dominates the second. Thus, the first alternative dominates the second in two steps. It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly, dominance can occur in three steps, four steps, and so on, the value of each obtained by raising the matrix to the corresponding power. The rank order of an alternative is the sum of the relative values for dominance in its row, in one step, two steps, and so on averaged over the number of steps. The question is whether this average tends to a meaningful limit.

We can think of the alternatives as the nodes of a directed graph. With every directed arc from node i to node j (which need not be distinct) is associated a nonnegative number a of the dominance matrix. In graph-theoretic terms this is the intensity of the arc. Define a k-walk to be a sequence of k arcs such that the terminating node of each arc except the last is the source node of the arc which succeeds it. The *intensity of a k-walk* is the product of the intensities of the arcs in the walk. With these ideas, we can interpret the matrix A: the (i, j) entry of A is the sum of the intensities of all k-walks from node i to node j [**2**, p. 203].

Definition. The dominance of an alternative along all walks of length $k \geqslant m$ is given by

$$\frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e}.$$

Observe that the entries of $A^k e$ are the row sums of A^k and that $e^T A^k e$ is the sum of all the entries of A.

THEOREM. *The dominance of each alternative along all walks k, as $k \to \infty$, is given by the solution of the eigenvalue problem $Aw = \lambda_{\max} w$.*

Proof. Let

$$s_k = \frac{A^k e}{e^T A^k e},$$

and

$$t_m = \frac{1}{m}\sum_{k=1}^{m} s_k.$$

The convergence of the components of t_m to the same limits as the components of s_m is the standard Cesaro summability and is shown in Hardy [**3**]. Since

$$s_k = \frac{A^k e}{e^T A^k e} \to w \text{ as } m \to \infty$$

[**5**, p. 171] where w is the normalized principal right eigenvector of A, we have

$$t_m = \frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e} \to w \text{ as } k \to \infty.$$

The essence of the principal eigenvector is to rank alternatives according to dominance in terms of walks. The well-known logarithmic least squares method (LLSM): find the vector $v = (v_1, \ldots, v_n)$ which minimizes the expression

$$\sum_{i,j=1}^{n}\left(\log a_{ij} - \log\frac{v_i}{v_j}\right)^2,$$

sometimes proposed as an alternative method of solution, obtains results which coincide with the principal eigenvector for matrices of order two and three, but deviate from it for higher order and can lead to rank reversal. As an example of rank reversal, compare the eigenvector solution and the LLSM solution for the 5×5 matrix given earlier:

Teacher	Eigenvector solution	LLSM solution
A	.0893	.0819
B	.3287	.3433
C	.1983	.2089
D	.3413	.3214
E	.0424	.0418

In the eigenvector solution the teachers are ranked in descending order D, B, C, A, E; whereas the LLSM solution ranks them B, D, C, A, E.

The LLSM minimizes deviations over all the entries of the matrix. The principal eigenvector does not attempt to minimize anything, but maximizes information preserved from all known relations of dominance.

References

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Presenting a Mathematics Play

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Have you ever thought of presenting a mathematics play? Imre Lakatos' *Proofs and Refutations* can be easily understood by freshmen, yet contains interesting and even challenging material for all levels, including teachers and research mathematicians. The ideas he presents about how mathematics is formed and how it should be taught, his witty presentation, and his interest in the history of mathematics make it worth staging. The play is really a revised version of the first chapter of Lakatos' 1961 Cambridge Ph.D. thesis in the Philosophy of Mathematics, and it has been very carefully crafted. In it, he confronts the classical picture of mathematical development as a steady accumulation of established truths. He shows that mathematics grew instead through a richer, more dramatic process of successive improvements of creative hypotheses, by attempts to prove them, and by criticism of these attempts: by proofs and refutations.

Set in a classroom with one ultrapatient teacher and at least a dozen *very* bright students (who unwittingly draw their opinions from Euler, Legendre, Cauchy, Möbius, and others), the play is very lively with a rich cast of characters, from dogmatists to skeptics. The topic under discussion