Motivation

Recall the Fourier transform of functions on $(\mathbb{R}, +)$: if $f : \mathbb{R} \to \mathbb{R}$, $\int_{\mathbb{R}} f^2 < \infty$, then the Fourier transform of f is the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(h) := \int_{\mathbb{R}} \exp(-ihx) f(x)\, dx$$

We want to define a similar transformation on (compact) groups. In this tutorial we study the Fourier transform on $\mathbb{S}_n$, the symmetric group on n elements.
There are three aspects of Fourier transform:

- Algebraic: in a sense, the Fourier transform preserves some important algebraic structures of the group. For instance, if we act on the group $(\mathbb{R}, +)$ by a left translation: $f'(x) = f(x-t)$, then this corresponds to a natural action on the Fourier transform of f: $\hat{f}'(h) = \exp^{-iht} \hat{f}(h)$. Or if we have convolution: $\widehat{f * g}(h) = \hat{f}(h)\hat{g}(h)$.
- Analytic: terms in the Fourier transform gives smoothness information on the function. This is important in signal processing.
- Algorithm: the efficiency of the Fast Fourier transform (FFT) makes it popular in practice.

Fourier transform on $\mathbb{S}_n$

Definition 1. A representation of a group G on a vector space V is a group homomorphism $\phi : G \to GL(V, \mathbb{F})$, where $GL(V, \mathbb{F})$ is the general linear group of a vector space V over the field $\mathbb{F}$.

When V is of dimension $d < \infty$ (which it is in our case), then we can identify $GL(V, \mathbb{F})$ with $GL_d(\mathbb{F})$, which is the space of invertible $d \times d$ matrices with entries in $\mathbb{F}$.
Example Let $G = \mathbb{S}_n$. Then $\rho : \mathbb{S}_n \to GL_d(\mathbb{F})$ is a representation of $\mathbb{S}_n$ if and only if ρ is a homomorphism:

$$\rho(\sigma_1\sigma_2) = \rho(\sigma_1)\rho(\sigma_2) \quad \text{for} \quad \sigma_1, \sigma_2 \in \mathbb{S}_n.$$

Example The exponential function $x \mapsto \exp(-ihx)$ is a representation of $(\mathbb{R}, +)$ on $GL_1(\mathbb{C})$.

This is the key in the usual Fourier transform. Note that h serves as an indexing over all possible representations of the group $(\mathbb{R}, +)$. Therefore, generalizing this idea, we define the Fourier transform for functions $f : \mathbb{S}_n \to \mathbb{C}$ as:

$$\hat{f}(\lambda) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma)\rho_\lambda(\sigma)$$

where λ (for the moment) serves as an 'indexing' parameter.

Definition 2. An irreducible representation of a group is a group representation that has no nontrivial invariant subspaces. Otherwise it is called reducible.

On a compact group G, reducible representations over $\mathbb{C}$ can be written as direct sum of irreducible representations. Hence we are interested in irreducible representations for $\mathbb{S}_n$. What are the possible representations on $\mathbb{S}_n$?

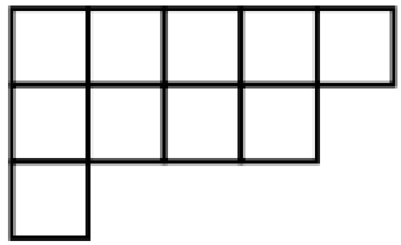

Figure 1. Young diagram (5,4,1)

Young diagram and representations of $\mathbb{S}_n$

Young diagram. Let $\{\lambda_i : i = 1 \ldots k\}$ be the cardinality of a partition of n objects into k boxes. In other words, $\lambda_i \in \mathbb{N}$, $\sum_{i=1}^{k} \lambda_i = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$. Arranging the boxes in a stack, the diagram obtained is called the Young diagram. An example of a Young diagram for the partition $5, 4, 1$ on 10 objects is included below. The boxes are filled with numbers from 1 to n, and the resulting table with entries is called a Young tableau. In a standard Young tableau, the entries increase from left to right, top to bottom. The dimension of a (standard) Young diagram is the number of distinct ways the boxes can be filled.

Young tableaux and representations of $\mathbb{S}_n$. . There is a one-to-one correspondence between Young diagrams and irreducible representations of the symmetric group $\mathbb{S}_n$ over $\mathbb{C}$. Let λ refers to a Young diagram. Therefore we can write

$$\hat{f}(\lambda) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma)\rho_\lambda(\sigma) \tag{1}$$

where ρ_λ denotes an irreducible representation of $\mathbb{S}_n$ that correspond to λ.

Given a Young diagram λ, how can we construct ρ_λ? In this tutorial we give the formula and an example on $\mathbb{S}_3$. We do not prove the construction. Interested readers can refer to *Group representations in probability and statistics* (Diaconis), or *the symmetric group: representations, combinatorial algorithms and symmetric functions* (Sagan).

Let d be the dimension of λ. Then ρ_λ maps $\mathbb{S}_n$ to $GL_d(\mathbb{C})$, therefore we can index the entries of the matrix $\rho_\lambda(\sigma)$ by distinct Young tableaux τ, τ' of λ. Furthermore, any $\sigma \in \mathbb{S}_n$ can be written as products of adjacent transpositions, which are of the form $(i, i+1)$. Therefore, it is sufficient to define $[\rho_\lambda(i, i+1)]_{\tau,\tau'}$. The Young's orthogonal representation is:

$$[\rho_\lambda(i, i+1)]_{\tau,\tau'} = \begin{cases} d_\tau^{-1}(i, i+1) & \text{if} \quad \tau = \tau' \\ \sqrt{1 - d_\tau^{-2}(i, i+1)} & \text{if} \quad \tau' = (i, i+1)(\tau) \\ 0 & \text{else} \end{cases}$$

where:

- d_τ is the number of steps it take to move i to $i+1$ where north and east movements (up and right) are taken as positive, and south and west movements (down and left) are taken as negative.
- $(i, i+1)(\tau)$ refers to a filling of λ obtained from τ by applying the transposition $(i, i+1)$ (swapping i an $i+1$).

Note that this results in a sparse, symmetric matrix.

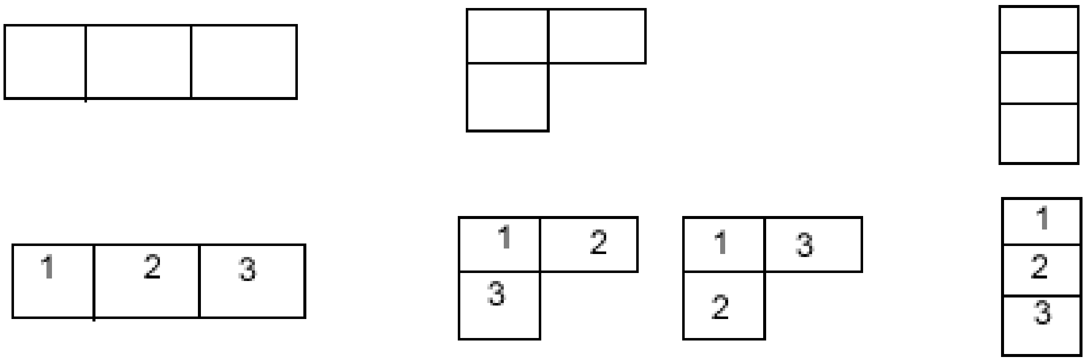

FIGURE 2. Young diagrams and Young tableaux for $\mathbb{S}_3$

Example on $\mathbb{S}_3$. There are 3 Young diagram on $n = 3$, and these are listed as unfilled boxes in the diagram below. Denote them $\lambda^1, \lambda^2, \lambda^3$ respectively. Note that $\rho_{\lambda^1}, \rho_{\lambda^3}$ are of dimension 1, and ρ_{λ^2} is of dimension 2. Let τ and τ' denote these two Young tableaux respectively. Then $\rho_{\lambda^2} : \mathbb{S}_3 \to G_2(\mathbb{C})$, and

$$\rho_{\lambda^2}((1,2)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \rho_{\lambda^2}((2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

ALGEBRAIC PROPERTIES OF THE FOURIER TRANSFORM ON $\mathbb{S}_n$

The Fourier transform on $\mathbb{S}_n$ defined in equation 1 satisfies the following properties:

- It is an invertible, norm-preserving transformation, where the norm of $f : \mathbb{S}_n \to \mathbb{C}$ is defined by
$$\|f\|^2 = \sum_{\sigma \in \mathbb{S}_n} |f(\sigma)|^2$$
and the norm of $\hat{f}$ is defined by
$$\|\hat{f}\| = \frac{1}{n!} \sum_{\lambda} d_\lambda \|\hat{f}_\lambda\|_F^2$$
where d_λ is the dimensionality of ρ_λ, and $\|\hat{f}_\lambda\|_F$ denotes the Frobenius norm of the matrix $\hat{f}_\lambda$.
- The inversion formula is
$$f(\sigma) = \frac{1}{n!} \sum_{\lambda} d_\lambda tr(\hat{f}(\lambda)(\rho_\lambda(\sigma))^{-1})$$
- Translation theorem: fix $\tau \in \mathbb{S}_n$. If $f^\tau(\sigma) = f(\tau^{-1}\sigma)$, then
$$\widehat{f^\tau}(\lambda) = \rho_\lambda(\tau)\hat{f}(\lambda)$$
- Convolution: let $(f * g)(\sigma) := \sum_\tau f(\sigma\tau^{-1})g(\tau)$. Then
$$\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$$
This is where we get computational gain.

Analytic viewpoint and connections to ranking

Let $\sigma \in \mathbb{S}_n$ denotes the ranking in which candidate i is ranked in position $\sigma(i)$. Define $f : \mathbb{S}_n \to \mathbb{R}$, $f(\sigma) =$ number of people voted for this ranking. Then the Fourier transform coefficients $\hat{f}(\lambda)$ gives 'smoothness' information of f. For example, the first term $\hat{f}((n)) = \sum_\sigma f(\sigma)$ gives the mean of the function. The first and second term $\hat{f}((n-1,1)) = \sum_{\sigma:\sigma(i)=j} f(\sigma)$ gives the number of votes for ranking i in position j (first order statistics). Inclusion of higher terms allow one to obtain higher order statistics.

Applications and references

On kernel computation:
R. Kondor and M. Barbosa: Ranking with kernels in Fourier space (COLT 2010):
http://www.its.caltech.edu/r̃isi/papers/KondorBarbosaCOLT10.pdf

Multi-object tracking:
R. Kondor, A. Howard and T. Jebara: Multi-object tracking with representations of the symmetric group (AISTATS 2007)
http://www.its.caltech.edu/r̃isi/papers/KondorHowardJebaraAISTATS07.pdf

Jonathan Huang, Carlos Guestrin, and Leonidas Guibas (2009): Fourier Theoretic Probabilistic Inference over Permutations. Journal of Machine Learning Research (JMLR), 10, 997-1070.
http://www.select.cs.cmu.edu/publications/paperdir/jmlr2009-huang-guestrin-guibas.pdf

Classical reference on the subject:
Diaconis: Group representations in probability and statistics, Lecture Notes 1988.
http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.lnms/1215467407