Motivation

Recall the Fourier transform of functions on $(\mathbb{R},+)$: if $f:\mathbb{R}\to\mathbb{R}$, $\int_{\mathbb{R}} f^2<\infty$, then the Fourier transform of f is the function $\hat{f}:\mathbb{R}\to\mathbb{R}$ defined by

$$\hat{f}(h) := \int_{\mathbb{R}} \exp(-ihx) f(x)\, dx$$

We want to define a similar transformation on (compact) groups. In this tutorial we study the Fourier transform on $\mathbb{S}_n$, the symmetric group on n elements.
There are three aspects of Fourier transform:

- Algebraic: in a sense, the Fourier transform preserves some important algebraic structures of the group. For instance, if we act on the group $(\mathbb{R},+)$ by a left translation: $f'(x) = f(x-t)$, then this corresponds to a natural action on the Fourier transform of f: $\hat{f}'(h) = \exp^{-iht}\hat{f}(h)$. Or if we have convolution: $\widehat{f*g}(h) = \hat{f}(h)\hat{g}(h)$.
- Analytic: terms in the Fourier transform gives smoothness information on the function. This is important in signal processing.
- Algorithm: the efficiency of the Fast Fourier transform (FFT) makes it popular in practice.

Fourier transform on $\mathbb{S}_n$

Definition 1. A representation of a group G on a vector space V is a group homomorphism $\phi: G\to GL(V,\mathbb{F})$, where $GL(V,\mathbb{F})$ is the general linear group of a vector space V over the field $\mathbb{F}$.

When V is of dimension $d<\infty$ (which it is in our case), then we can identify $GL(V,\mathbb{F})$ with $GL_d(\mathbb{F})$, which is the space of invertible $d\times d$ matrices with entries in $\mathbb{F}$.
Example Let $G=\mathbb{S}_n$. Then $\rho:\mathbb{S}_n\to GL_d(\mathbb{F})$ is a representation of $\mathbb{S}_n$ if and only if ρ is a homomorphism:

$$\rho(\sigma_1\sigma_2) = \rho(\sigma_1)\rho(\sigma_2) \quad \text{for} \quad \sigma_1,\sigma_2\in\mathbb{S}_n.$$

Example The exponential function $x\mapsto \exp(-ihx)$ is a representation of $(\mathbb{R},+)$ on $GL_1(\mathbb{C})$.

This is the key in the usual Fourier transform. Note that h serves as an indexing over all possible representations of the group $(\mathbb{R},+)$. Therefore, generalizing this idea, we define the Fourier transform for functions $f:\mathbb{S}_n\to\mathbb{C}$ as:

$$\hat{f}(\lambda) = \sum_{\sigma\in\mathbb{S}_n} f(\sigma)\rho_\lambda(\sigma)$$

where λ (for the moment) serves as an 'indexing' parameter.

Definition 2. An irreducible representation of a group is a group representation that has no nontrivial invariant subspaces. Otherwise it is called reducible.

On a compact group G, reducible representations over $\mathbb{C}$ can be written as direct sum of irreducible representations. Hence we are interested in irreducible representations for $\mathbb{S}_n$. What are the possible representations on $\mathbb{S}_n$?

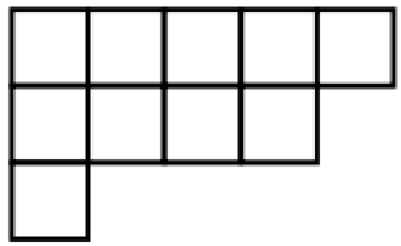

FIGURE 1. Young diagram (5,4,1)

YOUNG DIAGRAM AND REPRESENTATIONS OF $\mathbb{S}_n$

Young diagram. Let $\{\lambda_i : i = 1 \dots k\}$ be the cardinality of a partition of n objects into k boxes. In other words, $\lambda_i \in \mathbb{N}$, $\sum_{i=1}^{k} \lambda_i = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$. Arranging the boxes in a stack, the diagram obtained is called the Young diagram. An example of a Young diagram for the partition $5, 4, 1$ on 10 objects is included below. The boxes are filled with numbers from 1 to n, and the resulting table with entries is called a Young tableau. In a standard Young tableau, the entries increase from left to right, top to bottom. The dimension of a (standard) Young diagram is the number of distinct ways the boxes can be filled.

Young tableaux and representations of $\mathbb{S}_n$. . There is a one-to-one correspondence between Young diagrams and irreducible representations of the symmetric group $\mathbb{S}_n$ over $\mathbb{C}$. Let λ refers to a Young diagram. Therefore we can write

$$\hat{f}(\lambda) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma) \rho_\lambda(\sigma) \tag{1}$$

where ρ_λ denotes an irreducible representation of $\mathbb{S}_n$ that correspond to λ.

Given a Young diagram λ, how can we construct ρ_λ? In this tutorial we give the formula and an example on $\mathbb{S}_3$. We do not prove the construction. Interested readers can refer to *Group representations in probability and statistics* (Diaconis), or *the symmetric group: representations, combinatorial algorithms and symmetric functions* (Sagan).

Let d be the dimension of λ. Then ρ_λ maps $\mathbb{S}_n$ to $GL_d(\mathbb{C})$, therefore we can index the entries of the matrix $\rho_\lambda(\sigma)$ by distinct Young tableaux τ, τ' of λ. Furthermore, any $\sigma \in \mathbb{S}_n$ can be written as products of adjacent transpositions, which are of the form $(i, i+1)$. Therefore, it is sufficient to define $[\rho_\lambda(i, i+1)]_{\tau,\tau'}$. The Young's orthogonal representation is:

$$[\rho_\lambda(i, i+1)]_{\tau,\tau'} = \begin{cases} d_\tau^{-1}(i, i+1) & \text{if} \quad \tau = \tau' \\ \sqrt{1 - d_\tau^{-2}(i, i+1)} & \text{if} \quad \tau' = (i, i+1)(\tau) \\ 0 & \text{else} \end{cases}$$

where:

- d_τ is the number of steps it take to move i to $i+1$ where north and east movements (up and right) are taken as positive, and south and west movements (down and left) are taken as negative.
- $(i, i+1)(\tau)$ refers to a filling of λ obtained from τ by applying the transposition $(i, i+1)$ (swapping i an $i+1$).

Note that this results in a sparse, symmetric matrix.

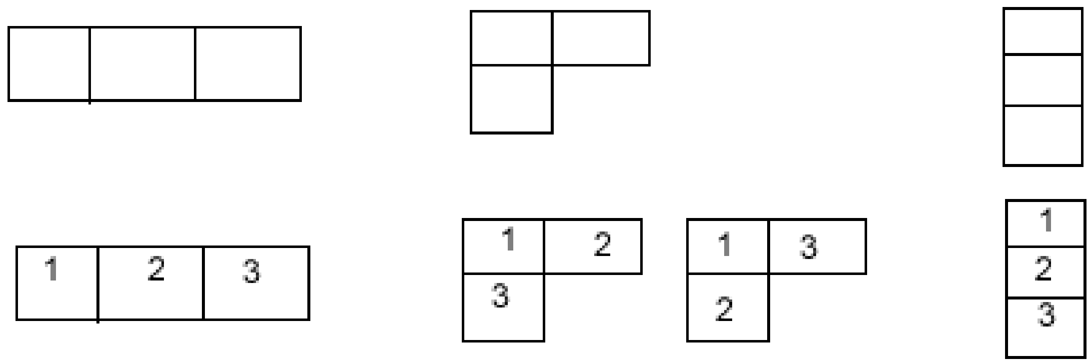

FIGURE 2. Young diagrams and Young tableaux for $\mathbb{S}_3$

Example on $\mathbb{S}_3$. There are 3 Young diagram on $n = 3$, and these are listed as unfilled boxes in the diagram below. Denote them $\lambda^1, \lambda^2, \lambda^3$ respectively. Note that $\rho_{\lambda^1}, \rho_{\lambda^3}$ are of dimension 1, and ρ_{λ^2} is of dimension 2. Let τ and τ' denote these two Young tableaux respectively. Then $\rho_{\lambda^2} : \mathbb{S}_3 \to G_2(\mathbb{C})$, and

$$\rho_{\lambda^2}((1,2)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \rho_{\lambda^2}((2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

ALGEBRAIC PROPERTIES OF THE FOURIER TRANSFORM ON $\mathbb{S}_n$

The Fourier transform on $\mathbb{S}_n$ defined in equation 1 satisfies the following properties:

- It is an invertible, norm-preserving transformation, where the norm of $f : \mathbb{S}_n \to \mathbb{C}$ is defined by
$$\|f\|^2 = \sum_{\sigma \in \mathbb{S}_n} |f(\sigma)|^2$$
and the norm of $\hat{f}$ is defined by
$$\|\hat{f}\| = \frac{1}{n!} \sum_{\lambda} d_\lambda \|\hat{f}_\lambda\|_F^2$$
where d_λ is the dimensionality of ρ_λ, and $\|\hat{f}_\lambda\|_F$ denotes the Frobenius norm of the matrix $\hat{f}_\lambda$.
- The inversion formula is
$$f(\sigma) = \frac{1}{n!} \sum_{\lambda} d_\lambda tr(\hat{f}(\lambda)(\rho_\lambda(\sigma))^{-1})$$
- Translation theorem: fix $\tau \in \mathbb{S}_n$. If $f^\tau(\sigma) = f(\tau^{-1}\sigma)$, then
$$\widehat{f^\tau}(\lambda) = \rho_\lambda(\tau)\hat{f}(\lambda)$$
- Convolution: let $(f * g)(\sigma) := \sum_\tau f(\sigma\tau^{-1})g(\tau)$. Then
$$\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$$
This is where we get computational gain.

Analytic viewpoint and connections to ranking

Let $\sigma \in \mathbb{S}_n$ denotes the ranking in which candidate i is ranked in position $\sigma(i)$. Define $f : \mathbb{S}_n \to \mathbb{R}$, $f(\sigma)$ = number of people voted for this ranking. Then the Fourier transform coefficients $\hat{f}(\lambda)$ gives 'smoothness' information of f. For example, the first term $\hat{f}((n)) = \sum_\sigma f(\sigma)$ gives the mean of the function. The first and second term $\hat{f}((n-1,1)) = \sum_{\sigma:\sigma(i)=j} f(\sigma)$ gives the number of votes for ranking i in position j (first order statistics). Inclusion of higher terms allow one to obtain higher order statistics.

An algebraic viewpoint

On the third day of the workshop, Michael Orrison continued the tutorial with an algebraic view of the Fourier transform, and how the Fourier transform on the symmetric group can be applied to voting. In this section we quickly sketches the concepts and results mentioned by Orrison. Detailed explanations with examples are in the following two papers:

M. Clausen and U. Baum
Fast Fourier Transforms For Symmetric Groups: Theory and Implementation
Mathematics of Computation, Vol 61(204), 833-847, 1993

Z. Daugherty, A. K. Eustis, G. Minton, and M. E. Orrison:
Voting, The Symmetric Group, and Representation Theory
Amer. Math. Monthly, 116 (2009), no. 8, pp. 667687

Let G be a finite group (for example, the symmetric group $\mathbb{S}_n$). The group algebra of G over $\mathbb{C}$ is defined as

$$\mathbb{C}G := \{f | f : G \to \mathbb{C}, (f+g)(x) = f(x) + g(x), (f * g)(x) = \sum_{y \in G} f(xy^{-1})g(y)\}$$

Then Wedderburn's theorem applied to $\mathbb{C}G$ states that

$$\mathbb{C}G \cong \oplus_{i=1}^{h} \mathbb{C}^{d_i \times d_i}.$$

That is, the group algebra of G over $\mathbb{C}$ is isomorphic to an algebra of block diagonal matrices with complex entries.

Definition 3. The discrete fourier transform for G is any algebra isomorphism that realizes the isomorphism implied in Wedderburn theorem. In other words, any isomorphism $D : \mathbb{C}G \to \oplus_{i=1}^{h} \mathbb{C}^{d_i \times d_i}$ is a discrete Fourier transform.

Note that this isomorphism is a coordinate map, hence it can be written as $D = \oplus_{i=1}^{h} D_i$. The D_i's are irreducible representations of $\mathbb{C}G$. In the context introduced previously, these are the irreducible ρ. From algebra we have the following facts:

- If G is abelian then $d_i = 1$ for all i since D is an isomorphism.
- The dimensions d_i's of the irreducible representations D_i's are unique. The number of irreducible representations h is the number of conjugacy classes of G.
- Since D is an isomorphism, as vector spaces, we have $\sum_i d_i^2 = |G|$

The Fast Fourier Transform (FFT) is an efficient algorithm for computing a suitable Fourier transform matrix for a given group. The case for the symmetric group was explored in section 2 of the paper by Clausen and Baum. As stated in the paper, a uniform approach to designing efficient DFT algorithms is based on adapting the irreducible representations to a chain of subgroups, since evaluation of D at a generic input vector $a \in \mathbb{C}G$ can be reduced to several evaluations of D at elements of $\mathbb{C}U$, where U is a subgroup of G. For the symmetric group, there is a natural chain: $\mathbb{S}_n > \mathbb{S}_{n-1} > \ldots > S_1$, and this leads to the FFT algorithm on symmetric groups. For further information, we refer the interested reader to the aforementioned paper, which has detailed examples, theorems and proofs, together with simulation results.

Applications and references

On kernel computation:
R. Kondor and M. Barbosa: Ranking with kernels in Fourier space (COLT 2010)

Multi-object tracking:
R. Kondor, A. Howard and T. Jebara: Multi-object tracking with representations of the symmetric group (AISTATS 2007)

Jonathan Huang, Carlos Guestrin, and Leonidas Guibas (2009): Fourier Theoretic Probabilistic Inference over Permutations. Journal of Machine Learning Research (JMLR), 10, 997-1070.

Classical reference on the subject:
Diaconis: Group representations in probability and statistics, Lecture Notes 1988.