Talk @ SIAM Applied A.G
Fort Collins, Co 2013

1) Notation : $X \subset \mathbb{P}(V^\vee)$: $\dim(V^\vee) = n+1$

X smooth : $X_d = \nu_d(X) \subset \mathbb{P}(\mathrm{Sym}^d(V)^\vee)$

ie: X_d is the image of X under the d-th Veronese embedding.

$\mathbb{P}_d = \nu_d(\mathbb{P}^n) \subset \mathbb{P}(\mathrm{Sym}^d(V)^\vee)$.

$\langle X \rangle$ = linear span of X

$$\sigma_s(X) = \overline{\bigcup_{p_i \in X} \langle p_1 \ldots p_s \rangle}$$

2) Results discussed are from

I-K Iarrobino-Kanev : Power Sums, Gorenstein Algebras & Determinantal Loci

B-G-L : Determinantal Equations

for Secant Varieties, and the

Eisenbud-Koh-Stillman Conjecture
(Buczynski-Ginensky-Landsberg)

& Buczynska-Buczynski B-B

Secant Varieties to high degree

Veronese reembeddings, Catalecticant

matrices & smoothable Gorenstein schemes

2A) I apologize for inexact or incorrect attributions

3) Classical Fact (Mumford) If $d \gg 0$

then $X_d \subset \langle X_d \rangle$ is defined by

(low rank) quadrics

3A) ~~Pf:~~ $\mathbb{P}_d$ ~~is defined by~~ ~~$\mathbb{P}_d$~~

3A) $\mathbb{P}_d \subset \mathbb{P}(S_{ym}{}^d(V)^\vee)$ is defined by (low rank) quadrics & if d is s.t. $I(X)$ is gen'd in degree $\leq d$ then $\nu_d(I(X))$ is generated by linear forms ~~are ~~ /neare late fine~~

$\langle X_d \rangle \subset \mathbb{P}(S_{ym}{}^d(V)^\vee)$. $X_d \subset \langle X_d \rangle$ is then defined by the quadrics defin'g $\mathbb{P}_d$

ie $X_d = \langle X_d \rangle \cap \mathbb{P}_d$.

4) So for secant varieties 2 question:

(A) is $\sigma_r(\mathbb{P}_d)$ defined by 'nice' eqns

(B) is $\sigma_r(X_d) = \langle X_d \rangle \cap \sigma_r(\mathbb{P}_d)$.

5) We only have set-theoretic results
Don't know about scheme ~~theory~~ theoretically

6) (B) was addressed in B-G-L

<u>Thm</u> (B-G-L) if $d \geq \mathrm{Gcl}(h_X)+r-1$ then set theoretically $\sigma_r(X_d) = \langle X_d\rangle \cap \sigma_r(R_d)$

6A) I expect this is true as a scheme theoretic statement + $\mathrm{Gcl}(h_X)$ is essentially the regularity of I_X.

7) A- is more difficult (not well kn

8) Sad Fact: (I-K, B-B) we don't know the equations of $\sigma_r(R_d)$.

9) Notation Dump

$$
\begin{tikzcd}
\mathbb{P}^n \arrow[r, "v_d"] \arrow[drr, "f_i"'] & \mathbb{P}(\mathrm{Sym}^d(V)^\vee) \arrow[r, hook, "F"] & \mathbb{P}((\mathrm{Sym}^i(V)\otimes \mathrm{Sym}^{d-i}(V))^\vee) \\
 & & S_i = \mathbb{P}(\mathrm{Sym}^i(V)^\vee) \times \mathbb{P}(\mathrm{Sym}^{d-i}(V)^\vee) \arrow[u, "\sigma_i"']
\end{tikzcd}
$$

9bis Then $\mathbb{P}_d \subset S_i$ ($\mathbb{P}_d$ is the rank one tensor in either $Sym^d(V)$ or $Sym^i(V) \otimes Sym^{d-i}(V)$).

& hence $\sigma_r(\mathbb{P}_d) \subset \sigma_r(S_i)$.

$\gamma_r^i = \sigma_r(S_i) \cap \mathbb{P}(Sym^d(V))$

(so $\gamma_r^{i,d-i}(V)$ b.d.t ...).

10) Secant varieties to Segre embeddings are understood - their eqns are the $(r+1) \times (r+1)$ minors of the catalecticant ~~matrix~~ matrix (z_{ij}) - where z_{ij} $x_i \otimes y_j$, w/ x_i running over a basis of $Sym^i(V)$ & y_j a basis of $Sym^{d-i}(V)$.

10a) eqns of γ_r^i are the 'nice' eqns of $\sigma_r(X)$.

11) When is $\sigma_r(X) = \gamma_r^i \cap \bigcap_i \gamma_r^r$?

12) $\mathcal{K}_r(X) = \bigcup_{R \in Hilb_r(X)} \langle R \rangle$ ← degree r zero dim'l schemes on X.

So $\sigma_r(X) \subset \mathcal{K}_r(X)$ since

$\sigma_r(X) = \bigcup_{R \in Hilb^0_r(X)} \langle R \rangle$ ← prin irred component of $Hilb^r(X)$

13) Technical fact. $\mathcal{K}_r(X) = \bigcup_{R \in Hilb^r(X) \,\&\, R\text{-Gor}} \langle R \rangle$

ie we can restrict to Gorenstein schemes

13a) ~~and~~ not hard - it's in B-G-L.

14) Main ~~technical~~ result: (B-B)

$d \geq 2r \quad r \leq c \leq d-r$ then

14bis) ~~$K_r(X) = \gamma_r^L$~~

$K_r(\mathbb{P}d) = \gamma_r^L$

14A) Thus the "obvious" eqns. define a variety that is generally bigger than $\sigma_r(X)$

14B) Result is more precise $p \in Sym^d(V)$ $p \in \gamma_r^L$ then we can explicitly construct a saturated Gorenstein ideal J s.t. $V(J)$ is a degree r - zero dim'l scheme w/ $p \in \langle V(J) \rangle$.

15) Result are proved using a polarity key idea (as I understand it). Given a Artinian Gorenstein alg R $\exists$ $p \in Sym^d(V)$ s.t. $R = Sym^*(V^*)/Ann(p)$

21) So program ~~is~~ amounts to finding "non-expected" eqns.

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Some Key Questions

① Can one find a new set of equations (class) that define $\sigma_2(X_d) \subset Sec_2(X_d)$ of $d \gg 0$

② Can the results of B-G-L & B-B be extended to be scheme theoretic

21'. In fact, one should think of both classes of results as syzygy statements. Can one extend the results (for d even larger) to the higher syzygies of $\sigma_2(X_d)$.

In particular if for $d \gg 0$ do we had

16) Since $\gamma_r^c = \mathcal{K}_r(X)$ & every R raised in $\sigma_r(X)$ is smoothable - key fact to hold
* Every zero-dim'l Gorenstein scheme is smoothable.

17) So $(*) \iff \mathcal{K}_r(X) = \sigma_r(X)$.

18-b) (B-B) $* \Rightarrow \mathcal{K}_r(X) = \sigma_r(X)$ & if $d \geq 2r-1$ & $\sigma_r(X_d) = \mathcal{K}_r(X_d)$ then $*$ holds.

18) $*$ holds if $n \leq 3$ or $r \leq 10$

19) $*$ known to fail if $n \geq 6$ & $r \geq 14$
$n = 5$ & $r \geq 42$ $\quad n = 4$ $r \geq 140$
(I-K + others)

20) In these cases

$$\sigma_r(X_d) \subsetneq \bigcap_L \gamma_r^c(X_d).$$

$K_{p,q}(X_d) = K_{p,q}(\mathbb{P}_d)$ in some range then perhaps combining Snowden's work w/ an answer to question #3 we can answer question #1. MORAL: #1 seems VERY HARD

#3) Is there an exact relationship between the syzygies of X & $\sigma_2(X)$

Specifically if ~~$d \gg 0$~~ $d \geq d(\Delta) = d(q)$ then the ~~nth~~ n-th syzygy module of X_d is the $(n-r)$ th syzygy module of $\sigma_2(X_d)$ for $d \geq d(h)$ &

$h \leq j$