## STAT 309: MATHEMATICAL COMPUTATIONS I <br> FALL 2022 <br> PROBLEM SET 4

1. Let $A \in \mathbb{R}^{m \times n}$ where $m \geq n$ and $\operatorname{rank}(A)=n$. Suppose GECP is performed on $A$ to get

$$
\Pi_{1} A \Pi_{2}=L U
$$

where $L \in \mathbb{R}^{m \times n}$ is unit lower triangular, $U \in \mathbb{R}^{n \times n}$ is upper triangular, and $\Pi_{1} \in \mathbb{R}^{m \times m}$, $\Pi_{2} \in \mathbb{R}^{n \times n}$ are permutation matrices.
(a) Show that $U$ is nonsingular and that $L$ is of the form

$$
L=\left[\begin{array}{l}
L_{1} \\
L_{2}
\end{array}\right]
$$

where $L_{1} \in \mathbb{R}^{n \times n}$ is nonsingular.
(b) We will see how the $L U$ factorization may be used to solve the least squares problem

$$
\min _{\mathbf{x} \in \mathbb{R}^{n}}\|A \mathbf{x}-\mathbf{b}\|_{2} .
$$

(i) Show that the problem may be solved via

$$
U \widetilde{\mathbf{x}}=\mathbf{y}, \quad L^{\top} L \mathbf{y}=L^{\top} \widetilde{\mathbf{b}},
$$

where $\widetilde{\mathbf{b}}=\Pi_{1} \mathbf{b}$ and $\widetilde{\mathbf{x}}=\Pi_{2}^{\top} \mathbf{x}$.
(ii) Describe how you would compute the solution $\mathbf{y}$ in

$$
L^{\top} L \mathbf{y}=L^{\top} \widetilde{\mathbf{b}} .
$$

2. Let $\varepsilon>0$. Consider the matrix

$$
A=\left[\begin{array}{cc}
1 & 1 \\
1 & 1+\varepsilon \\
1 & 1-\varepsilon
\end{array}\right]
$$

(a) Why is it a bad idea to solve the normal equation associated with $A$, i.e.

$$
A^{\top} A \mathbf{x}=A^{\top} \mathbf{b}
$$

when $\varepsilon$ is small?
(b) Show that the condensed $L U$ factorization of $A$ is

$$
A=L U=\left[\begin{array}{cc}
1 & 0 \\
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{ll}
1 & 1 \\
0 & \varepsilon
\end{array}\right]
$$

(c) Why is it a much better idea to solve the normal equation associated with $L$, i.e.

$$
L^{\top} L \mathbf{y}=L^{\top} \widetilde{\mathbf{b}} ?
$$

This shows that the method in Problem 1 is a more stable method than using the normal equation in (a) directly.
(d) Show that the Moore-Penrose pseudoinverse of $A$ is

$$
A^{\dagger}=\frac{1}{6}\left[\begin{array}{ccc}
2 & 2-3 \varepsilon^{-1} & 2+3 \varepsilon^{-1} \\
0 & 3 \varepsilon^{-1} & -3 \varepsilon^{-1}
\end{array}\right]
$$

(e) Describe a method to compute $A^{\dagger}$ given $L$ and $U$. Verify that your method is correct by checking it against the expression in (d).
3. We will now discuss an alternative method to solve the least squares problem in Problem 1 that is more efficient when $m-n<n$.
(a) Show that the least squares problem in Problem 1 is equivalent to

$$
\min _{\mathbf{z} \in \mathbb{R}^{n}}\left\|\left[\begin{array}{c}
I_{n} \\
S
\end{array}\right] \mathbf{z}-\widetilde{\mathbf{b}}\right\|_{2}
$$

where $S=L_{2} L_{1}^{-1}$ and $L_{1} \mathbf{y}=\mathbf{z}$. Here and below, $I_{n}$ denotes the $n \times n$ identity matrix.
(b) Write

$$
\widetilde{\mathbf{b}}=\left[\begin{array}{l}
\widetilde{\mathbf{b}}_{1} \\
\widetilde{\mathbf{b}}_{2}
\end{array}\right]
$$

where $\widetilde{\mathbf{b}}_{1} \in \mathbb{R}^{n}$ and $\widetilde{\mathbf{b}}_{2} \in \mathbb{R}^{m-n}$. Show that the solution $\mathbf{z}$ is given by

$$
\mathbf{z}=\widetilde{\mathbf{b}}_{1}+S^{\top}\left(I_{m-n}+S S^{\top}\right)^{-1}\left(\widetilde{\mathbf{b}}_{2}-S \widetilde{\mathbf{b}}_{1}\right)
$$

(c) Explain why when $m-n<n$, the method in (a) is much more efficient than the method in Problem 1. For example, what happens when $m=n+1$ ?
4. Let $\mathbf{c} \in \mathbb{R}^{n}$ and consider the linearly constrained least squares problem/minimum norm linear system

$$
\begin{aligned}
\operatorname{minimize} & \|\mathbf{w}\|_{2} \\
\text { subject to } & A^{\top} \mathbf{w}=\mathbf{c}
\end{aligned}
$$

(a) If we write $\widetilde{\mathbf{c}}=\Pi_{2}^{\top} \mathbf{c}$ and $\widetilde{\mathbf{w}}=\Pi_{1} \mathbf{w}$, show that

$$
\widetilde{\mathbf{w}}=L\left(L^{\top} L\right)^{-1} U^{-\top} \widetilde{\mathbf{c}}
$$

where $U^{-\top}=\left(U^{-1}\right)^{\top}=\left(U^{\top}\right)^{-1}$, a standard notation that we will also use below. (Hint: You'd need to use something that you've already determined in an earlier part).
(b) Write

$$
\widetilde{\mathbf{w}}=\left[\begin{array}{l}
\widetilde{\mathbf{w}}_{1} \\
\widetilde{\mathbf{w}}_{2}
\end{array}\right]
$$

where $\widetilde{\mathbf{w}}_{1} \in \mathbb{R}^{n}$ and $\widetilde{\mathbf{w}}_{2} \in \mathbb{R}^{m-n}$. Show that

$$
\widetilde{\mathbf{w}}_{1}=L_{1}^{-\top} U^{-\top} \widetilde{\mathbf{c}}-S^{\top} \widetilde{\mathbf{w}}_{2} .
$$

(c) Write $\mathbf{d}=L_{1}^{-\top} U^{-\top} \widetilde{\mathbf{c}}$. Deduce that $\widetilde{\mathbf{w}}_{2}$ may be obtained either as a solution to

$$
\min _{\widetilde{\mathbf{w}}_{2} \in \mathbb{R}^{m-n}}\left\|\left[\begin{array}{c}
S^{\top} \\
I_{m-n}
\end{array}\right] \widetilde{\mathbf{w}}_{2}-\left[\begin{array}{c}
\mathbf{d} \\
\mathbf{0}
\end{array}\right]\right\|_{2}
$$

or as

$$
\widetilde{\mathbf{w}}_{2}=\left(I_{m-n}+S S^{\top}\right)^{-1} S \mathbf{d} .
$$

Note that when $m-n<n$, this method is advantageous for the same reason in Problem 3.
5. So far we have assumed that $A$ has full column rank. Suppose now that $\operatorname{rank}(A)=r<$ $\min \{m, n\}$.
(a) Show that the $L U$ factorization obtained using GECP is of the form

$$
\Pi_{1} A \Pi_{2}=L U=\left[\begin{array}{l}
L_{1} \\
L_{2}
\end{array}\right]\left[\begin{array}{ll}
U_{1} & U_{2}
\end{array}\right]
$$

where $L_{1}, U_{1} \in \mathbb{R}^{r \times r}$ are triangular and nonsingular.
(b) Show that the above equation may be rewritten in the form

$$
\Pi_{1} A \Pi_{2}=\left[\begin{array}{c}
I_{r} \\
S_{1}
\end{array}\right] L_{1} U_{1}\left[\begin{array}{ll}
I_{r} & S_{2}^{\mathrm{T}}
\end{array}\right]
$$

for some matrices $S_{1}$ and $S_{2}$.
(c) Hence show that the Moore-Penrose inverse of $A$ is given by

$$
A^{\dagger}=\Pi_{2}\left[\begin{array}{ll}
I_{r} & S_{2}^{\top}
\end{array}\right]^{\dagger} U_{1}^{-1} L_{1}^{-1}\left[\begin{array}{c}
I_{r} \\
S_{1}
\end{array}\right]^{\dagger} \Pi_{1}
$$

(d) Using the general formula (derived in the lectures) for the Moore-Penrose inverse of a rank-retaining factorization, what do you get for $A^{\dagger}$ ?

