Empirical Processes of Dependent Random Variables

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EMPIRICAL PROCESSES OF DEPENDENT RANDOM VARIABLES

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Abstract: Empirical processes for stationary, causal sequences are considered. We establish empirical central limit theorems for classes of indicators of left half lines, absolutely continuous functions and piecewise differentiable functions. Sample path properties of empirical distribution functions are also discussed. The results are applied to linear processes and Markov chains.

1 Introduction

The theory of empirical processes plays a fundamental role in statistics and it has many applications ranging from parameter estimation to hypothesis testing (van der Vaart and Wellner, 1996). The literature of empirical processes for independent random variables is huge and there are many deep results; see Donsker (1952), Dudley (1978), Pollard (1984), Giné and Zinn (1984), Shorack and Wellner (1986), Ossiander (1987), van der Vaart and Wellner (1996).

To deal with random variables such as time series that are dependent, one naturally asks whether results obtained under the independence assumption remain valid. Such asymptotic theory is evidently useful for statistical inference of stochastic processes. Without the independence assumption, it is more challenging to develop a weak convergence theory for the associated empirical processes. One way out is to impose strong mixing conditions to ensure the asymptotic independence; see Billingsley (1968), Gastwirth and Rubin (1975), Withers (1975), Mehra and Rao (1975), Doukhan et al (1995), Andrews and Pollard (1994), Shao and Yu (1996), Rio (1998, 2000) and Pollard (2002) among others. Other special

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processes that have been discussed include linear processes and Gaussian processes; see Dehling and Taqqu (1989) and Csörgő and Mielniczuk (1996) for long and short-range dependent subordinated Gaussian processes and Ho and Hsing (1996) and Wu (2003a) for long-range dependent linear processes. A collection of recent results is presented in Dehling, Mikosch and Sorensen (2002). In that collection Dedecker and Louhichi (2002) made an important generalization of Ossiander’s (1987) result.

Here we investigate the empirical central limit problem for dependent random variables from another angle that avoids strong mixing conditions. In particular, we apply a martingale method and establish a weak convergence theory for stationary, causal processes. Our results are comparable with the theory for independent random variables in that the imposed moment conditions are optimal or almost optimal. We show that, if the process is short-range dependent in a certain sense, then the limiting behavior is similar to that of iid random variables in that the limiting distribution is a Gaussian process and the norming sequence is $\sqrt{n}$. For long-range dependent linear processes, one needs to apply asymptotic expansions to obtain $\sqrt{n}$-norming limit theorems (Section 6.2.2).

The paper is structured as follows. In Section 2 we introduce some mathematical preliminaries necessary for the weak convergence theory and illustrate the essence of our approach. Two types of empirical central limit theorems are established. Empirical processes indexed by indicators of left half lines, absolutely continuous functions, and piecewise differentiable functions are discussed in Sections 3, 4 and 5 respectively. Applications to linear processes and iterated random functions are made in Section 6. Section 7 presents some integral and maximal inequalities that may be of independent interest. Some proofs are given in Sections 8 and 9.

2 Preliminaries

Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary process. Denote by $F$ and $F_n(x) = n^{-1}\sum_{i=1}^{n} 1_{X_i \leq x}$, $x \in \mathbb{R}$, the marginal and empirical distribution functions. Let $\mathcal{G}$ be a class of measurable functions from $\mathbb{R}$ to $\mathbb{R}$. The centered $\mathcal{G}$-indexed empirical process is given by

\[
(P_n - P)g = \frac{1}{n} \sum_{i=1}^{n} g(X_i) - \mathbb{E}[g(X_1)] = \int_{\mathbb{R}} g(x) d[F_n(x) - F(x)], \quad g \in \mathcal{G}.
\]  (1)
The weak convergence theory concerns the limiting behavior of \( \{(P_n - P)g : g \in G\} \) under proper scaling. Our primary goal is to establish abstract Donsker theorems for dependent random variables. Namely, we aim at finding appropriate conditions on \((X_n)\) and \(G\) such that \( \{\sqrt{n}(P_n - P)g, g \in G\} \) converges in distribution to some tight Gaussian process \( W = \{W(g), g \in G\} \).

To describe the weak convergence theory, some mathematical apparatus is needed. Let the triple \((\Omega, \mathcal{B}(\Omega), \mathbb{P})\) be the probability space on which the process \((X_i)_{i \in \mathbb{Z}}\) is defined; let \(\ell^\infty(G)\) be the set of functions \(z : G \mapsto \mathbb{R}\) for which \(\|z\|_G := \sup_{g \in G} |z(g)| < \infty\). A random element \(\xi\) with values in \(\ell^\infty(G)\) is said to be tight if, for every \(\delta > 0\), there is a compact set \(K_\delta\) of \((\ell^\infty(G), \| \cdot \|_G)\) for which \(\mathbb{P}(\xi \not\in K_\delta) < \delta\). Assume that for any \(x \in \mathbb{R}\), sup\(g \in G |g(x) - Pg| < \infty\). Then \(\sqrt{n}(P_n - P)\) is a map from \(\Omega\) to \(\ell^\infty(G)\). We say that \(\sqrt{n}(P_n - P)\) converges weakly to the tight Gaussian process \(W\) if, for any bounded, continuous function \(h : (\ell^\infty(G), \| \cdot \|_G) \mapsto \mathbb{R}\), \(\lim_{n \to \infty} \mathbb{E}^*\{h[\sqrt{n}(P_n - P)]\} = \mathbb{E}[h(W)]\).

Here \(\mathbb{E}^*\) denotes the outer expectation: \(\mathbb{E}^* Z = \inf\{\mathbb{E}U : U \geq Z \text{ and } U \text{ is measurable}\}\). The outer probability of an arbitrary set \(A \subset \Omega\) is given by \(\mathbb{P}^*(A) = \mathbb{E}^*(1_A)\). The outer expectation is introduced to deal with the measurability issue which emerges when \(G\) is uncountable. See van der Vaart and Wellner (1996) for more on weak convergence theory.

It is well-known that \(\{\sqrt{n}(P_n - P)g, g \in G\} \) converges weakly to some tight Gaussian process \(\{W(g), g \in G\} \) if and only if the following two conditions are satisfied (Theorem 1.5.4, van der Vaart and Wellner, 1996):

(i) **Finite-dimensional convergence:** For any finite set of functions \(g_1, \ldots, g_k \in G\),

\[
[\sqrt{n}(P_n - P)g_1, \ldots, \sqrt{n}(P_n - P)g_k] \Rightarrow \text{a multivariate normal distribution.} \tag{2}
\]

(ii) **Tightness:** For every \(\delta, \eta > 0\), \(G\) can be partitioned into finitely many sets \(G_1, \ldots, G_p\) such that

\[
\limsup_{n \to \infty} \mathbb{P}^* \left[ \max_{1 \leq i \leq p} \sup_{g \in G_i} |\sqrt{n}(P_n - P)(g - h)| \geq \delta \right] \leq \eta. \tag{3}
\]

The finite-dimensional convergence (i) is trivial if \(X_k\) are iid and \(\mathbb{E}[g^2(X_1)] < \infty\). For dependent random variables, the central limit theorem (2) itself is an interesting and important problem and it has received much attention for more than a half century. Various strong mixing assumptions are imposed in early work [cf. Rosenblatt (1956), Ibragimov (1962), Bradley (2002), Doukhan (1994) and Peligrad (1996) among others]. Gordin (1969)

Generally speaking, it is more challenging to verify the tightness condition (ii). Consider the special case of the indicator function class $G = \{1_{s \leq s} : s \in \mathbb{R}\}$. Let $R_n(s) = \sqrt{n}[F_n(s) - F(s)]$. Then (3) requires that there exists a partition $\mathbb{R} = \bigcup_{i=1}^p T_i$ such that

$$
\limsup_{n \to \infty} \mathbb{P}\left[ \max_{1 \leq i \leq p} \sup_{s,t \in T_i} |R_n(s) - R_n(t)| \geq \delta \right] \leq \eta.
$$

The verification of the preceding relation is not simple partly due to the fact that $F_n$ is discontinuous. For iid random variables, there exist sophisticated tools such as chaining arguments and exponential inequalities. It is not straightforward to apply those tools to dependent random variables.

In our problem the major interest is to obtain comparable results without the iid assumption. It is necessary to impose certain structural assumptions on the underlying process $(X_n)_{n \in \mathbb{Z}}$ and the class $G$. In the early literature, strong mixing conditions have been imposed on $(X_n)$ [for example, Doukhan et al (1995), Pollard (2002)]. Here we restrict ourselves to causal processes. Let $(\varepsilon_k)_{k \in \mathbb{Z}}$ be independent and identically distributed (iid) random variables; let $J$ be a measurable function such that

$$
X_n = J(\ldots, \varepsilon_{n-1}, \varepsilon_n)
$$

is a proper random variable. Such processes are also known as causal Bernoulli shifts and they have received considerable attention recently; see Doukhan and Louhichi (1999), Prieur (2002), Doukhan (2003) among others. The framework (4) is general enough to allow many interesting and important examples. Prominent ones are linear processes and Markov chains arising from iterated random functions; see Section 6. For the class $G$, we consider indicators of left half lines, absolutely continuous functions and piecewise differentiable functions.

In this article we will apply the martingale method, and thus shed new light on this important problem. To illustrate the idea behind our approach, let $F_n = (\ldots, \varepsilon_{n-1}, \varepsilon_n)$
and denote by \( F_\varepsilon(x|F_n) = \mathbb{P}(X_{n+1} \leq x|F_n) \) the conditional distribution function of \( X_{n+1} \) given the sigma algebra \( \sigma(F_n) \). Assume throughout the paper that the conditional density \( f_\varepsilon(x|F_n) = (\partial/\partial x)F_\varepsilon(x|F_n) \) exists almost surely. Define the conditional empirical distribution function \( \tilde{F}_n(x) = n^{-1}\sum_{i=1}^n F_\varepsilon(x|F_i) \). Then

\[
F_n(x) - F(x) = [F_n(x) - \tilde{F}_n(x)] + [\tilde{F}_n(x) - F(x)].
\] (5)

The decomposition (5) has two important and useful properties. First, \( n[F_n(x) - \tilde{F}_n(x)] \) is a martingale with stationary, ergodic, and bounded martingale differences. Second, the function \( \tilde{F}_n - F \) is differentiable with derivative \( \tilde{f}_n(x) = f_\varepsilon(x|F_\varepsilon) \).

3 Empirical distribution functions

In this section we consider sample path properties and weak convergence of empirical distribution functions. Recall \( R_n(s) = \sqrt{n}[F_n(s) - F(s)] \). The classical Donsker theorem asserts that, if \( X_k, k \in \mathbb{Z}, \) are iid random variables, then \( \{R_n(s), s \in \mathbb{R}\} \) converges in distribution to an \( F \)-Brownian bridge process. The result has many applications in statistics. To understand the behavior at the two extremes \( s = \pm \infty \), we need to consider the weighted version \( \{R_n(s)W(s), s \in \mathbb{R}\} \), where \( W(s) \to \infty \) as \( s \to \pm \infty \). Clearly, if \( W \) is bounded, then by the continuous mapping theorem, the weak convergence of the weighted empirical processes follows from that of \( R_n \). The Chibisov-O’Reilly Theorem concerns weighted empirical processes of iid random variables. A detailed account can be found in Shorack and Wellner (1986, Section 11.5). The case of dependent random variables has
been far less studied. For strong mixing processes see Mehra and Rao (1975) and Shao and Yu (1996). Section 3.1 generalizes the Chibisov-O’Reilly Theorem to dependent random variables. Section 3.2 considers weighted modulus of continuity of $R_n$. Proofs of Theorems 1 and 2 are given in Section 8. Let the weight function $W$ be of the form $W(t) = (1 + |t|)^\delta$ for some $\delta \geq 0$.

### 3.1 A weak convergence result

Let $m$ be a measure on $\mathbb{R}$ and $T_n(\theta) = \sum_{i=1}^n h(\theta, \mathcal{F}_i) - n\mathbb{E}[h(\theta, \mathcal{F}_1)]$, where $h$ is a measurable function such that $\|h(\theta, \mathcal{F}_1)\| < \infty$ for almost all $\theta$, $(m)$. Define

$$\sigma(h, m) = \sum_{j=0}^{\infty} \sqrt{\int_{\mathbb{R}} \|P_0 h(\theta, \mathcal{F}_j)\|^2 m(d\theta)}. \quad (6)$$

Let $f'_\varepsilon(\theta|\mathcal{F}_k) = (\partial/\partial \theta)f_\varepsilon(\theta|\mathcal{F}_k)$. In the case that $h(\theta, \mathcal{F}_k) = f_\varepsilon(\theta|\mathcal{F}_k)$ or $h(\theta, \mathcal{F}_k) = f'_\varepsilon(\theta|\mathcal{F}_k)$, we write $\sigma(f_\varepsilon, m)$ or $\sigma(f'_\varepsilon, m)$ for $\sigma(h, m)$.

**Theorem 1.** Let $\gamma > 0$ and $q > 2$. Assume $\mathbb{E}(|X_1|^{\gamma}) < \infty$ and

$$\int_{\mathbb{R}} \mathbb{E}[f_\varepsilon^{q/2}(u|\mathcal{F}_0)] w_{\gamma-1+q/2}(du) < \infty. \quad (7)$$

In addition assume

$$\sigma(f_\varepsilon, w_{1+2\gamma/q}) + \sigma(f'_\varepsilon, w_{-1-2\gamma/q}) < \infty. \quad (8)$$

Then (i) 

$$\mathbb{E} \left[ \sup_{s \in \mathbb{R}} |R_n(s)|^2 (1 + |s|)^{2\gamma/q} \right] = O(1) \quad (9)$$

and (ii) the process $\{R_n(s)(1 + |s|)^{\gamma/q}, s \in \mathbb{R}\}$ converges weakly to a tight Gaussian process.

An important issue in applying Theorem 1 is the verification of (8), which is basically a short-range dependence condition. For many important models such as linear processes and Markov chains, (8) is easily verifiable; see Section 6. In particular, if $X_n$ is a linear process, then (8) reduces to the conventional definition of the short-range dependence of linear processes.
Remark 1. Let \( \alpha_n = \sup \{ \| \mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B) \| : A \in \mathcal{A}_0, B \in \mathcal{B}_n \} \) be the strong mixing coefficients, where \( \mathcal{A}_k = \sigma(X_i, i \leq k) \) and \( \mathcal{B}_n = \sigma(X_j, j \geq n) \). If \( X_n \) is strong mixing, namely \( \alpha_n \to 0 \), Rio (2000) showed that \( \mathbb{E}[\sup_{s \in \mathbb{R}} |R_n(s)|^2] \leq (1 + 4 \sum_{k=0}^{n-1} \alpha_k)(2 + \log n)^2 \); see Proposition 7.1 therein. Clearly the bound in (9) is sharper. \( \diamond \)

Corollary 1. Let \( \gamma \geq 0 \). Assume that \( \mathbb{E}(|X_1|^\gamma) < \infty \) and \( \sup_u f_\varepsilon(u|\mathcal{F}_0) \leq \tau \) hold for some \( \delta > 0 \) and \( \tau < \infty \). Further assume that \( \sigma(f_\varepsilon, w_{\gamma+\gamma'}) < \infty \) and \( \sigma(f'_\varepsilon, w_{-\gamma}) < \infty \). Then \( \{R_n(s)(1 + |s|)^{\gamma/2}, s \in \mathbb{R}\} \) converges weakly to a tight Gaussian process.

Proof. Let \( q = 2(\gamma + \delta + 1)/(\gamma + 1) \) and \( \gamma' = \gamma q/2 \). Then \( \gamma' - 1 + q/2 = \gamma + \delta \). We shall apply Theorem 1 with \( \gamma' \) and \( q \). Since \( \mathbb{E}[f_\varepsilon(u|\mathcal{F}_0)] = f(u) \), (7) follows from

\[
\int_{\mathbb{R}} \mathbb{E}[f_\varepsilon^{q/2}(u|\mathcal{F}_0)] w_{\gamma'-1+q/2}(du) \leq \tau^{q/2-1} \int_{\mathbb{R}} \mathbb{E}[f_\varepsilon(u|\mathcal{F}_0)] w_{\gamma'-1+q/2}(du) = \tau^{q/2-1} \mathbb{E}[(1 + |X_1|)^{\gamma+\delta}] < \infty.
\]

Note that \( 2\gamma'/q = \gamma \). Then (8) holds and the corollary follows from Theorem 1. \( \diamond \)

It is interesting to compare Corollary 1 with the Chibisov-O’Reilly Theorem, which concerns weighted empirical processes of iid random variables. The moment condition \( \mathbb{E}(|X_1|^\gamma) < \infty \) of Corollary 1 is almost necessary in the sense that it cannot be replaced by the weaker one

\[
\mathbb{E}\{ |X_1|^{\gamma} \log^{-1}(2 + |X_1|)(\log \log(10 + |X_1|))^{-\lambda} \} < \infty \tag{10}
\]

for some \( \lambda > 0 \). To see this let \( X_k \) be iid symmetric random variables with continuous, strictly increasing distribution function \( F \); let \( F^\# \) be the quantile function and \( m(u) = [1 + |F^\#(u)|]^{-\gamma/2} \). Then we have the distributional equality

\[
\{R_n(s)(1 + |s|)^{\gamma/2}, s \in \mathbb{R}\} =_{\mathbb{D}} \{R_n(F^\#(u))/m(u), u \in (0, 1)\},
\]

Assume that \( F(s)(1 + |s|)^{\gamma} \) is increasing on \((-\infty, G)\) for some \( G < 0 \). Then \( m(u)/\sqrt{u} \) is decreasing on \((0, F(G))\). By the Chibisov-O’Reilly Theorem, \( \{R_n(F^\#(u))/m(u), u \in (0, 1)\} \) is tight if and only if \( \lim_{t \to 0} m(t)/\sqrt{t \log \log(t^{-1})} = \infty \), namely

\[
\lim_{u \to -\infty} F(u)(1 + |u|)^{\gamma} \log \log |u| = 0. \tag{11}
\]
The above condition controls the heaviness of the tail of $X_1$. Let $F(u) = |u|^{-\gamma}(\log \log |u|)^{-1}$ for $u \leq -10$. Then it is easily seen that (10) holds, while (11) is violated. It is unclear whether stronger versions of (10) such as $\mathbb{E}(|X_1|^\gamma) < \infty$ or $\mathbb{E}[|X_1|^\gamma \log^{-1}(2 + |X_1|)] < \infty$ are sufficient.

### 3.2 Modulus of continuity

Theorem 2 below concerns the weighted modulus of continuity of $R_n(\cdot)$. Sample path properties of empirical distribution functions of iid random variables have been extensively explored; see for example Csörgő et al. (1986), Shorack and Wellner (1986) and Einmahl and Mason (1988) among others. It is far less studied for the dependent case.

**Theorem 2.** Assume $\mathbb{E}(|X_1|^\gamma) < \infty$ and $\sup_u f_\varepsilon(u|F_0) \leq \tau$ for some $\gamma \geq 0$ and $\tau < \infty$. Let $\delta_n < 1/2$ be a sequence of positive numbers such that $(\log n)^2/(\alpha - 2) = O(n\delta_n)$, where $2 < q < 4$. Further assume that, for some $\mu \leq 1$,

$$\sigma(f_\varepsilon, w_{2\gamma/q - \mu}) + \sigma(f_\varepsilon', w_{2\gamma/q + \mu}) < \infty. \quad (12)$$

Then there exists a constant $0 < C < \infty$, independent of $n$ and $\delta_n$, such that for all $n \geq 1$,

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} (1 + |t|)^{2\gamma/q} \sup_{|s| \leq \delta_n} |R_n(t + s) - R_n(t)|^2 \right] \leq C\delta_n^{1-2/q}. \quad (13)$$

### 4 Absolutely continuous functions

Let $\mathcal{AC}$ be the collection of all absolutely continuous functions $g : \mathbb{R} \mapsto \mathbb{R}$. In this section we shall investigate the behavior of the empirical process $\sqrt{n}(P_n - P)g$ indexed by $g \in \mathcal{G}_{\gamma,\mu}$ or $g \in \mathcal{H}_{\eta,\delta}$, where $\mathcal{G}_{\gamma,\mu}$, $\gamma \geq 0$, $\mu \leq 1$, is the weighted Sobolev class

$$\mathcal{G}_{\gamma,\mu} = \left\{ g \in \mathcal{AC} : \int_{\mathbb{R}} g^2(u)w_{-\gamma-\mu}(du) + \int_{\mathbb{R}} [g'(u)]^2w_{-\gamma+\mu}(du) \leq 1 \right\} \quad (14)$$

and the class $\mathcal{H}_{\eta,\delta}$, $\eta, \delta \geq 0$, $\eta - \delta \leq 1$, is given by

$$\mathcal{H}_{\eta,\delta} = \left\{ g \in \mathcal{AC} : |g(u)| \leq (1 + |u|)^\eta, |g'(u)| \leq (1 + |u|)^\delta \text{ for all } u \in \mathbb{R} \right\}. \quad (15)$$

Functions in $\mathcal{G}_{\gamma,\mu}$ can be unbounded if $\gamma + \mu > 1$. See Remark 2 for some properties of $\mathcal{G}_{\gamma,\mu}$ and $\mathcal{H}_{\eta,\delta}$. In the study of empirical central limit theorems for independent random
variables, bracketing conditions are often imposed. The class of differentiable functions is an important case that bracketing conditions can be verified; see Chapter 2.7 in van der Vaart and Wellner (1996). Functions considered in robust inference are often absolutely continuous.

**Theorem 3.** (i) Let $\gamma \geq 0$ and $\mu \leq 1$. Assume $E(|X_1|^\gamma) < \infty$ and
\[
\sigma(f, w_{\gamma+\mu}) < \infty.
\]
(16)
Then $\{\sqrt{n}(P_n - P)g : g \in \mathcal{G}_{\gamma,\mu}\}$ converges weakly to a tight Gaussian process $\{W(g) : g \in \mathcal{G}_{\gamma,\mu}\}$ with mean 0 and finite covariance function
\[
\text{cov}[W(g_1), W(g_2)] = \sum_{k \in \mathbb{Z}} \text{cov}[g_1(X_0), g_2(X_k)], \quad g_1, g_2 \in \mathcal{G}_{\gamma,\mu}.
\]
(ii) Let $\eta, \delta \geq 0$. Assume that
\[
\sum_{j=1}^{\infty} j^{\alpha\eta} \sqrt{\mathbb{P}[j^{\alpha} \leq |X_1| < (j+1)^{\alpha}]} < \infty
\]
if $\eta - \delta < 1$, where $\alpha = (1 + \delta - \eta)^{-1}$, and that
\[
\sum_{j=1}^{\infty} 2^{j\eta} \sqrt{\mathbb{P}(2^j \leq |X_1| < 2^{j+1})} < \infty
\]
if $\eta - \delta = 1$. In addition, let $m(dx) = (1 + |x|)^{1+2\eta} \log^2(2 + |x|)dx$ and assume
\[
\sigma(f, m) < \infty.
\]
(19)
Then the conclusion in (i) still holds with $\mathcal{G}_{\gamma,\mu}$ replaced by $\mathcal{H}_{\eta,\delta}$.

The proof of Theorem 3 is given in Section 4.1. Theorem 3 generalizes the empirical central limit theorems in Giné and Zinn (1986) in two ways: by allowing dependence and by considering wider classes. Giné and Zinn (1986) considered iid random variables and Lipschitz continuous functions. In particular, they show that (i) if $\eta = \delta = 0$ (the class of bounded Lipschitz continuous functions), then (17) is necessary and sufficient for the tightness of $\{\sqrt{n}(P_n - P)g : g \in \mathcal{H}_{0,0}\}$ and (ii) if $\delta = 0$, then (18) is necessary and sufficient for the tightness of $\{\sqrt{n}(P_n - P)g : g \in \mathcal{H}_{0,\cdot}\}$, where $\mathcal{H}_{0,\cdot} = \{g : |g'(u)| \leq 1\}$ is the class
of Lipschitz continuous functions. Further consideration is given in van der Vaart (1996). So in the cases of \( \eta = \delta = 0 \) and \( \delta = 0, \eta = 1 \), the conditions (17) and (18) are optimal.

The dependence structure of \( (X_k)_{k \in \mathbb{Z}} \) will inevitably find its way into the weak convergence theory. Here the assumption on the dependence is encapsulated as \( \sigma(h, m) < \infty \), which is a consequence of the requirement of \( \sqrt{n} \)-normalization of the partial sums in view of \( \int_{\mathbb{R}} \| T_n(\theta) \|^2 m(d\theta) \leq n\sigma^2(h, m) \) by Lemma 4. Conditions (16) and (19) actually imply more. They are natural conditions for the asymptotic normality of \( \sqrt{n}(P_n - P)g \) for a fixed \( g \in G_{\gamma, \mu} \) or \( g \in H_{\eta, \delta} \); see the proof of Theorem 3 in Section 4.1. If \( X_n \) is a long-range dependent linear process, then (16) and (19) are violated and the norming sequence of \( (P_n - P)g \) is different from \( \sqrt{n} \). In this case asymptotic expansions are needed to ensure a Gaussian limit process with a \( \sqrt{n} \)-normalization (cf. Section 6.2.2).

Doukhan et al (1995), Rio (1998) and Pollard (2002) considered stationary, absolutely regular processes. Bracketing conditions are given based on a metric which involves mixing coefficients. Rio (2000, Theorem 8.1) established an empirical central limit theorem for strong mixing processes indexed by Lipschitz continuous functions. In the case of causal processes, it is not easy to verify that they are absolutely regular or strong mixing. For linear processes to be strong mixing, quite restrictive conditions are needed on the decay rate of the coefficients (Doukhan, 1994). In comparison, Theorem 6 and Corollary 2 (cf. Section 6.2.1) impose a sharp condition on the coefficients of linear processes.

Some new weak dependence conditions are used in Doukhan and Louhichi (1999), Prieur (2002) and Dedecker and Prieur (2003b). Here we compare our results with theirs by applying them to the Gaussian process \( X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i} \), where \( a_i = (i+1)^{-\beta}, \beta > 1 \), and \( \varepsilon_n \) are iid standard normal. As mentioned in the preceding paragraph, Theorem 6 and Corollary 2 impose a sharp condition \( \beta > 1 \). Proposition 2 in Doukhan and Louhichi (1999) asserts that \( R_n \) is tight if

\[
\sup_{f \in \mathcal{I}} |\text{cov}(f(X_{t_1})f(X_{t_2}), f(X_{t_3})f(X_{t_4}))| = O(n^{-5/2-\delta})
\]

for some \( \delta > 0 \), where \( \mathcal{I} = \{ x \to \mathbf{1}_{s < x < t} : s, t \in \mathbb{R} \}, 0 \leq t_1 < t_2 < t_3 < t_4 \) and \( n = t_3 - t_2 \). Let \( t_1 = t_2 = 0, t_3 = t_4 = n \) and \( f(x) = \mathbf{1}_{x < 0} \). Elementary manipulations show that the covariance \( r(n) = \mathbb{E}(X_0X_n) \sim C_\beta n^{1-2\beta} \) for some constant \( 0 < C_\beta < \infty \) and \( |\mathbb{P}(X_0 < 0, X_n < 0) - \mathbb{P}(X_0 < 0)\mathbb{P}(X_n < 0)| = O(r(n)) \). So (20) requires \( 1 - 2\beta \leq -5/2 - \delta \), namely \( \beta > 7/4 \). The empirical central limit theorem in Dedecker and Prieur (2003b) is
not applicable to the process $X_n$. Actually, Corollary 4 in the latter paper assumes that $\sum_{k=1}^{\infty} \phi(k) < \infty$, where $\phi(k) = \sup_{t \in \mathbb{R}} \|P(X_k \leq t \mid F_0) - P(X_k \leq t)\|_\infty$. It is easily seen that $\phi(k) \equiv 1$ for the process $X_k$. By Theorem 1 in Prieur (2002), $R_n$ converges weakly if $\mathbb{E}|X_n - \sum_{i=0}^{n} a_i \xi_{n-i}| = O(n^{-2-2\sqrt{2} - \delta})$ for some $\delta > 0$. The latter condition implies $\beta > 2.5 + 2\sqrt{2}$. On the other hand, the existence of the conditional density $f_\varepsilon(x \mid F_n)$ is not assumed in those papers, while it is needed in our results.

Remark 2. We say that two classes $G_1$ and $G_2$ are equivalent, denoted by $G_1 \sim G_2$, if there is a constant $\lambda$ such that $\lambda^{-1}G_1 \subset G_2 \subset \lambda G_1$. In this case the process $\{\sqrt{n}(P_n - P)g : g \in G_1\}$ converges weakly if and only if $\{\sqrt{n}(P_n - P)g : g \in G_2\}$ does.

If $\eta \geq 1 + \delta$, then $\mathcal{H}_{\eta,\delta} \sim \mathcal{H}_{\delta+1,\delta}$. Clearly $\mathcal{H}_{\delta+1,\delta} \subset \mathcal{H}_{\eta,\delta}$. Let $u \geq 0$. Then

$$|g(u)| \leq |g(0)| + \int_0^u |g'(t)| dt \leq 1 + \int_0^u (1 + t)^\delta dt \leq (1 + u)^{1+\delta}$$

and hence $\mathcal{H}_{\eta,\delta} \subset \mathcal{H}_{\delta+1,\delta}$. The two classes $G_{\gamma,\mu}$ and $\mathcal{H}_{\eta,\delta}$ are closely related. If $\gamma > 1 + \eta + \delta$, then $\mathcal{H}_{\eta,\delta} \subset C_{\gamma,\eta,\delta}G_{\gamma,\eta-\delta}$ for some $C_{\gamma,\eta,\delta} < \infty$. There is no converse inclusion.

A particularly interesting case is when $\gamma > 0$ and $\mu = 1$. Let

$$G_{\gamma,\mu,0}^* = \{g \in AC : g(0) = 0 \text{ and } \int_{\mathbb{R}} [g'(u)]^2 w_{-\gamma,\mu}(du) \leq 1\}$$

(21)

and $G_{\gamma,\mu,0} = \{g \in G_{\gamma,\mu} : g(0) = 0\}$. By Hardy’s inequality (55), $G_{\gamma,1,0} \subset (1 + 4\gamma^{-2})^{1/2}G_{\gamma,1,0}$. The other relation $G_{\gamma,1,0}^* \subset G_{\gamma,1,0}$ is obvious. Therefore $G_{\gamma,1,0}^* \sim G_{\gamma,1,0}$. By Lemma 3, for $g \in G_{\gamma,\mu}$, $\sup_x g^2(x)(1 + |x|)^{-\gamma} \leq C_{\gamma,\mu}$ for some $C_{\gamma,\mu} < \infty$. So $g^2(0) \leq C_{\gamma,\mu}$ and there exists a constant $C'_{\gamma,\mu}$ such that $g - g(0) \in C_{\gamma,\mu}G_{\gamma,\mu}$ whenever $g \in G_{\gamma,\mu}$. Therefore $G_{\gamma,1,0} \sim \{g - g(0) : g \in G_{\gamma,1}\}$. Notice that $(P_n - P)(g - g(0)) = (P_n - P)g$. So the weak convergence problem of $\{\sqrt{n}(P_n - P)g : g \in G_{\gamma,1}\}$ is equivalent to the seemingly simpler one $\{\sqrt{n}(P_n - P)g : g \in G_{\gamma,1}^*\}$. \hfill \Box

4.1 Proof of Theorem 3.

We first consider the case $G = G_{\gamma,\mu}$. Following the ideas behind the decomposition (5), write $\sqrt{n}(P_n - P)g = M_n(g) + N_n(g)$, where

$$M_n(g) = n^{1/2} \int_{\mathbb{R}} g(x) d[F_n(x) - \tilde{F}_n(x)]$$

and

$$N_n(g) = n^{1/2} \int_{\mathbb{R}} g(x) [\tilde{f}_n(x) - f(x)] dx.$$

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The tightness of the process \( \{ \sqrt{n}(P_n - P)g : g \in G_{\gamma,\mu} \} \) follows from that of \( \{ M_n(g) : g \in G_{\gamma,\mu} \} \) and \( \{ N_n(g) : g \in G_{\gamma,\mu} \} \), which are asserted by Propositions 1 and 2 (cf. Sections 4.2 and 4.3) respectively. It remains to establish the finite-dimensional convergence. By Lemma 3, there exists \( C_{\gamma,\mu} < \infty \) such that

\[
\sup_{x \in \mathbb{R}} \frac{g^2(x)}{(1 + |x|)\gamma} \leq C_{\gamma,\mu} \int_{\mathbb{R}} g^2(u)w_{-\gamma-\mu}(du) + C_{\gamma,\mu} \int_{\mathbb{R}} [g'(u)]^2 w_{-\gamma+\mu}(du) \leq C_{\gamma,\mu}.
\]

Hence \( \|g(X_1)\|^2 \leq C_{\gamma,\eta} \mathbb{E}[|X_1 + 1|\gamma] \) is summable. Notice that for \( n \geq 1 \),

\[
P_0 g(X_n) = P_0 \mathbb{E}[g(X_n)|F_{n-1}] = P_0 \int_{\mathbb{R}} g(x)f_\varepsilon(x|F_{n-1})dx = \int_{\mathbb{R}} g(x)P_0f_\varepsilon(x|F_{n-1})dx.
\]

By the Cauchy-Schwarz inequality,

\[
\|P_0 g(X_n)\| \leq \sqrt{\int_{\mathbb{R}} g^2(x)w_{-\gamma-\mu}(dx)} \sqrt{\int_{\mathbb{R}} \|P_0 f_\varepsilon(x|F_{n-1})\|^2 w_{\gamma+\mu}(dx)},
\]

which in view of (16) is summable. By Lemma 1, the finite-dimensional convergence holds.

The other case that \( G = H_{\eta,\delta} \) similarly follows. For \( g \in H_{\eta,\delta} \), \( |g(u)| \leq (1 + |u|^\eta) \). By (17) or (18), we have \( \mathbb{E}(|X_1|^{2\eta}) < \infty \), which by Lemma 1 implies the finite-dimensional convergence. The tightness follows from (ii) of Proposition 1 and Proposition 2.

\[\diamondsuit\]

## 4.2 Tightness of \( M_n \)

In this section we establish the weak convergence of \( M_n \). The tightness of \( M_n \) involves moment conditions on \( X_1 \) and sizes of \( G_{\gamma,\mu} \) or \( H_{\eta,\delta} \), which are characterized by the parameters \((\gamma, \mu)\) and \((\eta, \delta)\). There is a tradeoff between the moment conditions and the sizes of the classes: larger classes require stronger moment conditions.

Interestingly, the tightness of \( M_n \) does not involve the dependence structure of \((X_k)_{k \in \mathbb{Z}}\).

In Section 6.2.2, we apply the result to strongly dependent processes.

**Proposition 1.** (i) Assume that \( \mathbb{E}(|X_1|^\gamma) < \infty \). Then the process \( \{ M_n(g) : g \in G_{\gamma,\mu} \} \) is tight. Consequently, it converges weakly to a tight Gaussian process \( \{ W_M(g) : g \in G_{\gamma,\mu} \} \) with mean 0 and covariance function

\[
\text{cov}[W_M(g_1), W_M(g_2)] = \mathbb{E}\{[P_0 g_1(X_1)][P_0 g_2(X_1)]\}, \quad g_1, g_2 \in G_{\gamma,\mu}.
\]

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(ii) Let \( \eta, \delta \geq 0 \). Assume (17) if \( \eta - \delta < 1 \) and (18) if \( \eta - \delta = 1 \). Then \( \{M_n(g) : g \in \mathcal{H}_{\eta,\delta}\} \) converges weakly to a centered, tight Gaussian process \( \{W_M(g) : g \in \mathcal{H}_{\eta,\delta}\} \) with the covariance function (22).


For a class \( G \) and a norm \( d \) let \( N(\delta, G, d) \) be the minimum number of \( \delta \)-brackets needed to cover \( G \). Here for two functions \( l \) and \( u \), the bracket \([l, u]\) is defined by \([l, u] = \{g \in G : l(x) \leq g(x) \leq u(x) \text{ for all } x\}\). If \( d(u-l) \leq \delta \), then we say that \([l, u]\) is a \( \delta \)-bracket. For a function \( g \) define its essential supremum \( d_2 \) norm by

\[
d_2^2(g) = \text{ess sup} \mathbb{E}[g^2(X_1)|\mathcal{F}_0].
\]  

Dedecker and Louhichi (2002) show that the \( \ell^\infty \)-valued map \( \sqrt{n}(P_n - P) \) is asymptotically \( d_2 \)-equicontinuous if

\[
\int_0^1 \sqrt{\log N(u, G, d_2)} du < \infty.
\]  

However, the norm \( d_2 \) in (23) is so strong that (24) is violated for many important applications. For example, let \( X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i} \), where \( \varepsilon_k \) are iid random variables. Assume that the support of \( \varepsilon_1 \) is the whole real line, \( a_0 = 1 \) and that there are infinitely many non-zero coefficients. Let \( \mathcal{G}_H = \{g(\cdot - \theta), \theta \in \mathbb{R}\} \), where \( g(x) = \max[-1, \min(x, 1)] \) is the derivative of a Huber function. Elementary calculations show that, since \( Y_0 = X_1 - \varepsilon_1 \) has the support of the whole real line, the bracket number \( N(\delta, \mathcal{G}_H, d_2) = \infty \) for all \( 0 < \delta < 1/2 \). Nevertheless, their generalization is quite important since the condition (24) is tractable in many cases. We apply their result in the proof of Proposition 1 and overcome the limitation by using a truncation argument.

**Remark 3.** Let \( \mathcal{G}_{DL} \) be a class of functions such that it satisfies Dedecker and Louhichi’s bracketing condition (24) and \( (\mathcal{G}_{DL}, d_2) \) is totally bounded. Then it is easily seen that the pairwise sum \( \mathcal{G}_{DL} + \mathcal{G}_{\gamma,\mu} = \{g + h : g \in \mathcal{G}_{DL}, h \in \mathcal{G}_{\gamma,\mu}\} \) is also a Donsker class, namely Proposition 1 holds for this pairwise sum.
4.3 Tightness of $N_n$

The tightness of $N_n$ requires the short-range dependence condition (16) or (19). The fact that $\tilde{F}_n$ is differentiable is quite useful.

**Proposition 2.** Assume (16) (resp. (19)). Then the process $\{N_n(g) : g \in G_{\gamma,\mu}\}$ (resp. $\{N_n(g) : g \in H_{\eta,\delta}\}$) is tight.

**Proof.** Recall $\{g \mathbb{1}_{|\cdot| \leq r} : g \in G_{\gamma,\mu}\}$. For $g \in G_{\gamma,\mu}$,

$$\left[\int_{|x| \geq r} |g(x)| |\tilde{f}_n(x) - f(x)| dx \right]^2 \leq \int_{|x| \geq r} g^2(x) w_{\gamma-\mu}(dx) \int_{|x| \geq r} |\tilde{f}_n(x) - f(x)|^2 w_{\gamma+\mu}(dx)$$

$$\leq \int_{|x| \geq r} |\tilde{f}_n(x) - f(x)|^2 w_{\gamma+\mu}(dx).$$

Applying Lemma 4 with $A = (-\infty, -r) \cup (r, \infty)$ and $T_n(\theta) = \sum_{i=1}^n f_\varepsilon(\theta|F_j)$, we have

$$E^* \left\{ \sup_{g \in G_{\gamma,\mu}} \left[ \sqrt{n} \int_{|x| \geq r} |g(x)| |\tilde{f}_n(x) - f(x)| dx \right]^2 \right\}$$

$$\leq nE \left\{ \int_{|x| > r} |\tilde{f}_n(x) - f(x)|^2 w_{\gamma+\mu}(dx) \right\}$$

$$\leq \left[ \sum_{j=0}^\infty \int_{|x| > r} \|P_0 f_\varepsilon(\theta|F_j)\|^2 w_{\gamma+\mu}(d\theta) \right]^2$$

which converges to 0 as $r \to \infty$. As in the proof of Proposition 1 (cf. Section 9), it remains to show that for fixed $r$, $\{N_n(g) : g \in 1_{|\cdot| \leq r} G_{\gamma,\mu}\}$ is tight. To this end, we apply Lemma 4 with $A = \mathbb{R}$ and $T_n(\theta) = \sum_{i=1}^n f_\varepsilon(\theta|F_j)$. By (16), there exists $\kappa < \infty$ such that

$$n \int_{\mathbb{R}} \|\tilde{f}_n(x) - f(x)\|^2 w_{\gamma+\mu}(dx) \leq \kappa$$

holds for all $n \in \mathbb{N}$. So the asymptotic equi-continuity follows from

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P^* \left\{ \sup_{\|g-h\|_\infty \leq \delta, g,h \in 1_{|\cdot| \leq r} G_{\gamma,\mu}} |N_n(g - h)| > \epsilon \right\}$$

$$\leq \lim_{\delta \to 0} \limsup_{n \to \infty} P \left\{ \sqrt{n} \int_{-r}^r |\tilde{f}_n(x) - f(x)| dx > \frac{\epsilon}{\delta} \right\}$$

$$\leq \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^2 \epsilon^2 nE \left[ \int_{-r}^r |\tilde{f}_n(x) - f(x)| w_{\gamma+\mu}(dx) \right]^2 = 0.$$
It is easily seen that \( (1_{|x| \leq r} g(x), \| \cdot \|_\infty) \) is totally bounded [cf. (87) and (88)]. Therefore, the process \( \{ N_n(g) : g \in G_{\gamma, \mu} \} \) is tight.

The other case that \( g \in H_{\eta, \delta} \) can be similarly proved by noticing that

\[
\sup_{g \in H_{\eta, \delta}} \int_A |g(x)\tilde{f}_n(x) - f(x)|w_n(dx) \leq \left( \int_A |\tilde{f}_n(x) - f(x)|^2m(dx) \right)^{1/2} \times \left( \int_A (1 + |x|)^{-1} \log^{-2}(2 + |x|)dx \right)^{1/2}
\]

in view of the Cauchy-Schwarz inequality.

\[ \Box \]

### 4.4 Finite-dimensional convergence

**Lemma 1.** Let \( g_1, \ldots , g_k \) be measurable functions such that \( g(X_i) \in L^2 \) and \( \mathbb{E}[g_i(X_1)] = 0, \) \( 1 \leq i \leq k. \) Let \( g = (g_1, \ldots , g_k) \). Assume

\[
\sum_{m=0}^{\infty} \| P_0 g_i(X_m) \| < \infty, \quad 1 \leq i \leq k.
\]  \hspace{1cm} (25)

Then \( \xi_{g_i} := \sum_{m=0}^{\infty} P_0 g_i(X_m) \in L^2 \) and

\[
n^{1/2}(P_n - P)g = n^{-1/2} \sum_{l=1}^{n} \{ g(X_l) - \mathbb{E}[g(X_1)] \} \Rightarrow N(0, \Sigma)
\]  \hspace{1cm} (26)

where \( \Sigma_{i,j} = \mathbb{E}(\xi_{g_i} \xi_{g_j}) = \sum_{l \in \mathbb{Z}} \text{cov}[g_i(X_l), g_j(X_l)] \).

**Proof.** The case in which \( k = 1 \) easily follows from Hannan (1973, Theorem 1) and Woodroofe (1992). For \( k \geq 2 \), we apply the Cramér-Wold device. Let \( \lambda_1, \lambda_2 \) be two real numbers. Using the relation \( \xi_{g_i} = \sum_{m=0}^{\infty} P_0 g_i(X_m) \), we have \( \lambda_1 \xi_{g_1} + \lambda_2 \xi_{g_2} = \xi_{\lambda_1 g_1 + \lambda_2 g_2} \), from which (26) easily follows with the stated covariance function.

\[ \Box \]

There are many other forms of central limit theorems for dependent random variables. Lemma 1 imposes simple and easily verifiable conditions. In addition, it also provides a very natural vehicle for the finite-dimensional convergence of \( \{ M_n(g) : g \in G_{\gamma, \mu} \} \); see the proof of Theorem 3 in Section 4.1.

Gordin (1969) obtained a general central limit theorem. However, as pointed out by Hall and Heyde (1980, p. 130), the condition imposed in Gordin (1969) is very difficult to check. To overcome the difficulty, in their book Hall and Heyde proposed Theorem 5.3 (see p. 133),
which states that the central limit theorem holds provided \( \sum_{n=0}^{\infty} P g(X_n) \) converges to some random variable \( \xi \) in \( L^2 \) with \( \|\xi\| > 0 \) and \( \lim_{n \to \infty} n \| (P_n - P) g \|^2 \to \|\xi\|^2 \). Due to the dependence, the verification of the latter condition could be difficult as well. In Lemma 1 we do not need to verify the latter condition. Another related result is given in Dedecker and Rio (2000). A key condition in the latter paper is that \( \sum_{i=0}^{\infty} g(X_0) \mathbb{E} [g(X_0) | \mathcal{F}_0] \) converges in \( L^1 \). Consider the linear process \( g(X_n) = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i} \), where \( a_i = (i+1)^{-\beta} \), \( \beta > 1/2 \) and \( \varepsilon_i \) are iid standard normal. Then their \( L^1 \) convergence condition requires \( \beta > 3/2 \). In comparison, (25) only needs \( \beta > 1 \).

5 Piecewise differentiable functions

Using the weak convergence theory for step and absolutely continuous functions, we can easily deal with piecewise differentiable functions. Let \( I \geq 1 \) be a fixed integer and \( \mathcal{G}_i = \{ \sum_{i=1}^{I} g_i(\cdot) 1_{\leq \theta_i} : g_i \in \mathcal{G}_{\gamma,1}, \ \theta_i \in \mathbb{R} \} \).

Theorem 4. Let \( \gamma > 0 \). Assume that \( \mathbb{E} (|X_1|^{\gamma + \delta}) < \infty \) and \( \sup_u f_\varepsilon (u | \mathcal{F}_0) \leq \tau \) hold for some \( \delta > 0 \) and \( \tau < \infty \). Further assume that \( \sigma(f_\varepsilon, w_{1+\gamma}) < \infty \) and \( \sigma(f_\varepsilon, w_{-1-\gamma}) < \infty \). Then \( \{ \sqrt{n}(P_n - P) g : g \in \mathcal{G}_i \} \) converges weakly to a Gaussian process.

Proof. Without loss of generality let \( I = 1 \). For all \( \theta \in \mathbb{R} \), the function \( g_\theta(\cdot) = g(\cdot) 1_{\leq \theta} + g(\theta) 1_{> \theta} \) is in \( \mathcal{G}_{\gamma,1} \) for some constant \( C_\gamma < \infty \). By Theorem 3, under the proposed conditions, \( \{ \sqrt{n}(P_n - P) g : g \in \mathcal{G}_{\gamma,1} \} \) converges weakly. It then suffices to show that \( \{ g(\theta) \sqrt{n}(P_n - P) 1_{> \theta} : g \in \mathcal{G}_{\gamma,1}, \ \theta \in \mathbb{R} \} \) is tight. Recall \( R_n(s) = \sqrt{n}[F_n(s) - F(s)] \). Notice that \( (P_n - P) 1_{> \theta} = -(P_n - P) 1_{\leq \theta} \). By Lemma 3, \( g^2(\theta) \leq C_\gamma (1 + |\theta|)^\gamma \). Hence

\[
\{ g(\theta) \sqrt{n}(P_n - P) 1_{> \theta} : g \in \mathcal{G}_{\gamma,1}, \ \theta \in \mathbb{R} \} \subset \{ \lambda R_n(s) : \lambda^2 \leq C_\gamma (1 + |s|)^\gamma, \ s \in \mathbb{R} \}.
\]

The latter is a process indexed by both \( \lambda \) and \( s \). By (76) and (79),

\[
\lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{E} \left\{ \sup_{|s| \geq r} \sup_{|\lambda| \leq C_\gamma^{1/2}(1+|s|)^{\gamma/2}} [\lambda^2 R_n^2(s)] \right\} = 0. \tag{27}
\]

Let \( \Gamma_r = \{ (\lambda, s) : \lambda^2 \leq C_\gamma (1 + |s|)^\gamma, \ |s| \leq r \} \) and \( \Gamma = \Gamma_\infty \). For \( (\lambda_1, s_1), (\lambda_2, s_2) \in \Gamma_r \),

\[
|\lambda_1 R_n(s_1) - \lambda_2 R_n(s_2)| \leq |\lambda_1 - \lambda_2| |R_n(s_1)| + |\lambda_2| |R_n(s_1) - R_n(s_2)| \leq |\lambda_1 - \lambda_2| \sup_{u \in \mathbb{R}} |R_n(u)| + C_{r,\gamma} |R_n(s_1) - R_n(s_2)|.
\]
Since \( \sup_{u \in \mathbb{R}} |R_n(u)| = O(1) \) and \( R_n(\cdot) \) is tight, it is easily seen that \( \{ \lambda R_n(s) : (\lambda, s) \in \Gamma_r \} \) is also tight. By (27), the process \( \{ \lambda R_n(s) : (\lambda, s) \in \Gamma \} \) is tight. Notice that

\[
\| \mathcal{P}_0 g(X_n) 1_{X_n \leq \theta} \| \leq \| \mathcal{P}_0 g_0(X_n) \| + |g(\theta)| \| \mathcal{P}_0 1_{X_n \leq \theta} \|
\]

is summable, the finite-dimensional convergence follows from Lemma 1.  

\[\diamond\]

6 Applications

Recall (6) for the definition of \( \sigma(h, m) \). To apply Theorems 1, 2 and 3, one needs to verify the short-range dependence condition that \( \sigma(h, m) \) is finite. In many important applications including Markov chains and linear processes, there is an \( \sigma(F_n) \)-measurable random variable \( Y_n \) such that

\[
P(X_{n+1} \leq x|F_n) = P(X_{n+1} \leq x|Y_n).
\]

Write \( Y_n = I(\ldots, \varepsilon_{n-1}, \varepsilon_n) \) and \( Y_n^* = I(\ldots, \varepsilon_{-1}, \varepsilon_0^*, \varepsilon_1, \ldots, \varepsilon_n) \), where \( (\varepsilon_i^*)_{i \in \mathbb{Z}} \) is an iid copy of \( (\varepsilon_i)_{i \in \mathbb{Z}} \), and \( h(\theta, F_n) = h(\theta, Y_n) \). It turns out that \( \sigma(h, m) \) is closely related to a weighted distance between \( Y_n \) and \( Y_n^* \). Define

\[
H_m(y) = \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} h(\theta, y) \right|^2 m(d\theta).
\]

**Proposition 3.** Let \( \rho_m(a, b) = |\int_a^b H_m^{1/2}(y)dy| \). Then for \( n \geq 0 \),

\[
\int_{\mathbb{R}} \| \mathcal{P}_0 h(\theta, Y_n) \|^2 m(d\theta) \leq \| \rho_m(Y_n, Y_n^*) \|^2.
\]

Hence we have \( \sigma(h, m) < \infty \) if

\[
\sum_{n=0}^{\infty} \| \rho_m(Y_n, Y_n^*) \| < \infty.
\]

**Proof.** Observe that \( E[h(\theta, Y_n)|\mathcal{F}_{-1}] = E[h(\theta, Y_n^*)|\mathcal{F}_{-1}] = E[h(\theta, Y_n^*)|\mathcal{F}_0] \). Then we have \( \mathcal{P}_0 h(\theta, Y_n) = E[h(\theta, Y_n) - h(\theta, Y_n^*)|\mathcal{F}_0] \). By the Cauchy-Schwarz inequality,

\[
\| \mathcal{P}_0 h(\theta, Y_n) \|^2 \leq \| h(\theta, Y_n) - h(\theta, Y_n^*) \|^2 \leq E \left[ \int_{Y_n} \left| \frac{\partial}{\partial y} h(\theta, y) \right| dy \right]^2.
\]
Let \( \lambda(y) = \sqrt{H_m(y)} \). Again by the Cauchy-Schwarz inequality,
\[
\int_{\mathbb{R}} \| P_0 h(\theta, Y_n) \|^2 m(d\theta) \leq \int_{\mathbb{R}} E \left[ \int_{Y_n} Y_n^* \lambda(y) dy \right] \int_{\mathbb{R}} E \left[ \int_{Y_n} Y_n^* \lambda(y) dy \right] m(d\theta)
\]
\[
= \int_{\mathbb{R}} E \left[ \int_{Y_n} Y_n^* \lambda(y) dy \right] \int_{\mathbb{R}} E \left[ \int_{Y_n} Y_n^* \lambda(y) dy \right] m(d\theta)
\]
So (30) follows.

Let \( h(\theta, Y_n) = f_{\varepsilon}(\theta | Y_n) \). Then \( H_m(y) \) can be interpreted as a measure of "local dependence" of \( X_{n+1} \) on \( Y_n \) at \( y \). If \( X_{n+1} \) and \( Y_n \) are independent, then \( f_{\varepsilon}(\theta | y) \) does not depend on \( y \), and hence \( H_m = 0 \). Let
\[
A(y; \delta) = \int_{\mathbb{R}} [f_{\varepsilon}(\theta | y) - f_{\varepsilon}(\theta | y + \delta)]^2 m(d\theta).
\]
Then \( \sqrt{A(y; \delta)} \) is the weighted \( L^2 \) distance between the conditional densities of the conditional distributions \( [X_{n+1}|Y_n = y] \) and \( [X_{n+1}|Y_n = y + \delta] \). Under suitable regularity conditions, \( \lim_{\delta \to 0} \delta^{-2} A(y; \delta) = H_m(y) \). Therefore the function \( H_m \) quantifies the dependence of \( X_{n+1} \) on \( Y_n \). Intuitively, (31) suggests short-range dependence in the sense that if we change \( \varepsilon_0 \) in \( Y_n \) to \( \varepsilon_0^* \), then the cumulative impact of the corresponding changes in \( Y_n \), measured by the weighted distance \( \rho_m \), is finite.

\textbf{Remark 4.} The random variable \( Y_n^* \) can be viewed as a coupled version of \( Y_n \). The coupling method is popular. See Section 3.1 in Dedecker and Prieur (2003a) and Section 4 in their (2003b) for some recent contributions.

\section{Iterated random functions}

Many nonlinear time series models assume the form of iterated random functions [Elton (1990), Diaconis and Freedman (1999)]. Let
\[
X_n = R(X_{n-1}, \varepsilon_n),
\]
where \( \varepsilon, \varepsilon_k, k \in \mathbb{Z} \), are iid random elements and \( R(\cdot, \cdot) \) is a bivariate measurable function. For the process (32), due to the Markovian property, (28) is satisfied with \( Y_n = X_n \). The
existence of stationary distribution of (32) has been widely studied and there are many versions of sufficient conditions; see Diaconis and Freedman (1999), Meyn and Tweedie (1994), Steinsaltz (1999), Jarner and Tweedie (2001), Wu and Shao (2004) among others. Here we adopt the simple condition proposed by Diaconis and Freedman (1999). Let the Lipschitz constant
\[ L_\varepsilon = \sup_{x \neq x'} \frac{|R(x, \varepsilon) - R(x', \varepsilon)|}{|x - x'|}. \]
Assume that there exist \( \alpha > 0 \) and \( x_0 \) such that
\[ E[L_\varepsilon^\alpha + |x_0 - R(x_0, \varepsilon)|^\alpha] < \infty \] and \( E[\log(L_\varepsilon)] < 0. \] (33)
Then there is a unique stationary distribution (Diaconis and Freedman, 1999). The latter paper also gives a convergence rate of an arbitrary initial distribution towards the stationary distribution. Wu and Woodroofe (2000) pointed out that the simple sufficient condition (33) also implies the geometric-moment contraction: There exist \( \beta > 0 \), \( r \in (0, 1) \) and \( C < \infty \) such that
\[ E[|F(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) - F(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_n)|^\beta] \leq Cr^n \] holds for all \( n \in \mathbb{N} \), where \( (\varepsilon'_k)_{k \in \mathbb{Z}} \) is an iid copy of \( (\varepsilon_k)_{k \in \mathbb{Z}} \). Hsing and Wu (2004) and Wu and Shao (2004) argued that (34) is a convenient condition to establish limit theorems. Dedecker and Prieur (2003a, b) discussed the relationship between (34) and some new dependence coefficients. Recently Douc et al (2004) considered subgeometric rates of convergence of Markov chains. In their paper they adopted total variational distance, while (34) involves the Euclidean distance.

Bae and Levental (1995) considered empirical central limit theorems for stationary Markov chains. It seems that the conditions imposed in their paper are formidable restrictive. Consider the Markov chain \( X_{n+1} = aX_n + \varepsilon_{n+1} \), where \( 0 < |a| < 1 \) and \( \varepsilon_i \) are iid standard normal random variables. Then the transition density is \( q(u|v) = (2\pi)^{-1/2} \exp\left[-(u - av)^2/2\right] \) and the chain has the stationary distribution \( N[0, (1 - a^2)^{-1}] \). Let \( \alpha(u) = (2\pi)^{-1/2} \left(1 - a^2\right)^{1/2} \exp\left[-u^2(1 - a^2)/2\right] \) be the marginal density. It is easily seen that \( \sup_{u, v \in \mathbb{R}} [q(u|v)/\alpha(u)] = \infty \). Conditions (1.1) and (1.2) in Bae and Levental (1995) require that this quantity is finite. For this process, by Theorem 6, \( \{\sqrt{n}(P_n - P)g \in G_{\gamma, \mu}\} \) converges weakly to a tight Gaussian process for any \( \gamma \geq 0 \) and \( \mu \leq 1 \).
6.1.1 AR models with ARCH Errors

Autoregressive models with conditional heteroscedasticity (ARCH) have received considerable attention over the last two decades. For recent work see Berkes and Horváth (2004) and Straumann and Mikosch (2003), where some statistical inference problems of such models are considered. Here we consider the model

\[ X_n = \alpha X_{n-1} + \varepsilon_n \sqrt{a^2 + b^2 X_{n-1}^2}, \tag{35} \]

where \((\varepsilon_i)_{i \in \mathbb{Z}}\) are iid random variables. In the case \(b = 0\), then (35) is reduced to the classical AR(1) model. If \(b \neq 0\), then the conditional variance of \(X_n\) given \(X_{n-1}\) is not a constant and the model is said to be heteroscedastic. We assume without loss of generality that \(b = 1\) since otherwise we can introduce \(a' = a/b\) and \(\varepsilon'_i = \varepsilon_i/b\). Let \(R(x, \varepsilon) = \alpha x + \varepsilon \sqrt{a^2 + x^2}\). Then \(L_\varepsilon \leq \sup_x |\partial R(x, \varepsilon)/\partial x| \leq |\alpha| + |\varepsilon|\). Assume \(r = \mathbb{E}[(|\alpha| + |\varepsilon|)^\beta] < 1\) for some \(\beta > 0\). Then (33) holds and there is a unique stationary distribution. It is easily seen that (34) holds with this \(\beta\) and \(r\), and \(\mathbb{E}(|X_1|^\beta) < \infty\).

We now compute \(H_{w_{\alpha}}(y)\). Denote by \(f_\varepsilon\) the density function of \(\varepsilon_i\). Write \(u = u(\theta, y) = (\theta - \alpha y)/\sqrt{a^2 + y^2}\). Then the conditional (transition) density \(f_\varepsilon(\theta|y) = (a^2 + y^2)^{-1/2}f_\varepsilon(u)\) and

\[ \frac{\partial}{\partial y} f_\varepsilon(\theta|y) = -\frac{y f_\varepsilon(u)}{(a^2 + y^2)^{3/2}} - \frac{\alpha f'_\varepsilon(u)}{a^2 + y^2} + \frac{(\theta - \alpha y) y f'_\varepsilon(u)}{(a^2 + y^2)^2}. \tag{36} \]

Assume that \(\kappa_\gamma := \int_{\mathbb{R}} f_\varepsilon^2(t) w_\gamma(dt) + \int_{\mathbb{R}} |f'_\varepsilon(t)|^2 w_{\gamma+2}(dt) < \infty\). Elementary calculations show that

\[ \int_{\mathbb{R}} f_\varepsilon^2(u(\theta, y)) w_\gamma(d\theta) = \int_{\mathbb{R}} f_\varepsilon^2 \left( \frac{v}{\sqrt{a^2 + y^2}} \right) (1 + |v + \alpha y|)^\gamma dv \leq C_\gamma (1 + |\alpha y|)^\gamma (a^2 + y^2)^{1/2} \kappa_\gamma + C_\gamma (a^2 + y^2)^{(1+\gamma)/2} \kappa_\gamma, \]

\[ \int_{\mathbb{R}} |f'_\varepsilon(u(\theta, y))|^2 w_\gamma(d\theta) \leq C_\gamma (1 + |\alpha y|)^\gamma (a^2 + y^2)^{1/2} \kappa_\gamma + C_\gamma (a^2 + y^2)^{(1+\gamma)/2} \kappa_\gamma \]

and

\[ \int_{\mathbb{R}} (\theta - \alpha y)^2 |f'_\varepsilon(u)|^2 w_\gamma(d\theta) \leq C_\gamma (1 + |\alpha y|)^\gamma (a^2 + y^2)^{3/2} \kappa_\gamma + C_\gamma (a^2 + y^2)^{(3+\gamma)/2} \kappa_\gamma. \]

Combining the preceding three inequalities, we have by (36) that

\[ H_{w_\gamma}(y) \leq \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} f_\varepsilon(\theta|y) \right|^2 w_\gamma(dy) \leq C_\alpha \gamma (1 + |y|)^{-3+\gamma}. \tag{37} \]
Theorem 5. Let $\mathbb{E}[(|\alpha| + |\varepsilon|)^\gamma] < 1$ for some $\gamma > 0$ and
\[
\int_{\mathbb{R}} f^2_{\varepsilon}(t)w_{\gamma+1}(dt) + \int_{\mathbb{R}} |f'_{\varepsilon}(t)|^2w_{\gamma+3}(dt) < \infty.
\] (38)

Then there exists $\phi \in (0, 1)$ such that $\rho_{w_{\gamma+1}}(Y_n, Y_n^*) = O(\phi^n)$ and hence \{\sqrt{n}(P_n-P)g : g \in G_{\gamma,1}\} converges weakly to a tight Gaussian process.

Proof. Let $r = \mathbb{E}[(|\alpha| + |\varepsilon|)^\gamma]$. By (34), $\mathbb{E}(|Y_n - Y_n^*|^\gamma) \leq Cr^n$ for some constant $C$. By (37), $H_{w_{\gamma+1}}(y) \leq C(1 + |y|)^{\gamma-2}$. Let $\lambda = \gamma/2$. If $0 < \lambda \leq 1$, by Lemma 2,
\[
\rho_{\gamma+1}(Y_n, Y_n^*) = O\left(\left\| \int_{Y_n} (1 + |y|)^{\lambda-1}dy \right\| \right) = O(\mathbb{E}(|Y_n - Y_n^*|^{2\lambda}))^{1/2} = O(r^{n/2}).
\]

If $\lambda > 1$, let $p = \lambda/(\lambda - 1)$. By Hölder’s inequality,
\[
\mathbb{E}[(1 + |Y_n|)^{2(\lambda-1)}|Y_n - Y_n^*|^2] \leq \{\mathbb{E}[(1 + |Y_n|)^{2(\lambda-1)p}]\}^{1/p} \times \{\mathbb{E}[|Y_n - Y_n^*|^{2\lambda}]\}^{1/\lambda} = O((r^n)^{1/\lambda}).
\]

Hence
\[
\left\| \int_{Y_n} (1 + |y|)^{\lambda-1}dy \right\|^2 \leq \left\| (1 + |Y_n|)^{\lambda-1} + (1 + |Y_n^*|)^{\lambda-1} \right\| |Y_n - Y_n^*|^2 \leq 4\mathbb{E}[(1 + |Y_n|)^{2(\lambda-1)}|Y_n - Y_n^*|^2] = O((r^n)^{1/\lambda}).
\]

Let $\phi = \max(r^{1/2}, r^{1/(2\lambda)}) < 1$. Then $\rho_{\gamma+1}(Y_n, Y_n^*) = O(\phi^n)$, and the theorem follows from (31), Proposition 3 and (i) of Theorem 3. \hfill \Box

Lemma 2. Let $0 < \lambda \leq 1$. Then for all $u, v \in \mathbb{R}$, $|\int_v^u (1 + |y|)^{\lambda-1}dy| \leq 2^{1-\lambda}|u - v|^\lambda/\lambda$.

Proof. It suffices to consider two cases (i) $u \geq v \geq 0$ and (ii) $u \geq 0 \geq v$. For case (i),
\[
\int_v^u (1 + |y|)^{\lambda-1}dy = \frac{1}{\lambda}[(1 + u)^\lambda - (1 + v)^\lambda] \leq \frac{1}{\lambda}(u - v)^\lambda.
\]

For the latter case, let $t = u - v$. Then
\[
\int_v^u (1 + |y|)^{\lambda-1}dy = \frac{1}{\lambda}[(1 + u)^\lambda - 1 + (1 + |v|)^\lambda - 1] \leq \frac{1}{\lambda}[2(1 + t/2)^\lambda - 2] \leq \frac{2^{1-\lambda}}{\lambda}t^\lambda
\]
and the lemma follows. \hfill \Box

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6.2 Linear processes

Let \( X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i} \), where \( \varepsilon_k, k \in \mathbb{Z} \), are iid random variables with mean 0 and finite and positive variance, and the coefficients \((a_i)_{i \geq 0}\) satisfy \( \sum_{i=0}^{\infty} a_i^2 < \infty \). Assume without loss of generality that \( a_0 = 1 \). Let \( F_\varepsilon \) and \( f_\varepsilon = F_\varepsilon' \) be the distribution and density functions of \( \varepsilon_k \). Then the conditional density of \( X_{n+1} \) given \( F_n \) is \( f_\varepsilon(x - Y_n) \) and (28) is satisfied, where \( Y_n = X_{n+1} - \varepsilon_{n+1} \). Limit theorems for short and long-range dependent linear processes are presented in Sections 6.2.1 and 6.2.2 respectively.

6.2.1 Short-memory linear processes

**Proposition 4.** Let \( \gamma \geq 0 \). Assume \( \mathbb{E}(|\varepsilon_k|^{2+\gamma}) < \infty \) and

\[
\kappa := \int_{\mathbb{R}} |f'_\varepsilon(u)|^2 w_\gamma(du) < \infty.
\]  

Then \( \rho_{w_\gamma}(Y_n, Y^*_n) = \mathcal{O}(|a_{n+1}|) \).

**Proof.** Using the elementary inequality \( 1 + |v + y| \leq (1 + |v|)(1 + |y|) \), we have

\[
H_{w_\gamma}(y) = \int_{\mathbb{R}} |f'_\varepsilon(u - y)|^2(1 + |u|)^\gamma du \\
\leq (1 + |y|)^\gamma \int_{\mathbb{R}} |f'_\varepsilon(v)|^2 w_\gamma(dv) = \kappa(1 + |y|)^\gamma
\]

and

\[
\left| \int_{Y_n}^{Y^*_n} \sqrt{H_{w_\gamma}(y)} dy \right| \leq \sqrt{\kappa} \left| \int_{Y_n}^{Y^*_n} (1 + |y|)^{\gamma/2} dy \right| \\
\leq \sqrt{\kappa}[(1 + |Y_n|)^{\gamma/2} + (1 + |Y^*_n|)^{\gamma/2}]|Y_n - Y^*_n|.
\]

Observe that \( Y_n \) and \( Y^*_n \) are identically distributed. Then

\[
\rho_{w_\gamma}(Y_n, Y^*_n) \leq 2\sqrt{\kappa}|| (1 + |Y_n|)^{\gamma/2} |Y_n - Y^*_n|| \\
= 2\sqrt{\kappa}|| a_{n+1} || (1 + |Y_n|)^{\gamma/2} (\varepsilon_0 - \varepsilon'_0) || \\
\leq 2\sqrt{\kappa}|| a_{n+1} || || (1 + |Y_n|)^{\gamma/2} \varepsilon_0 || + || (1 + |Y_n|)^{\gamma/2} \varepsilon'_0 ||.
\]

Since \( \mathbb{E}(|\varepsilon_k|^{2+\gamma}) < \infty \), it is easily seen that \( || (1 + |Y_n|)^{\gamma/2} \varepsilon'_0 || = \mathcal{O}(1) \) and \( || (1 + |Y_n|)^{\gamma/2} \varepsilon_0 || = \mathcal{O}(1) \). So the proposition follows. \( \diamond \)
Remark 5. By Proposition 4, if \( \gamma = \mu = 0 \), \( \varepsilon_k \) has second moment and \( \int_{\mathbb{R}} |f'(t)|^2 dt < \infty \), then (16) is equivalent to
\[
\sum_{n=1}^{\infty} |a_n| < \infty, \quad (40)
\]
which is a well-known condition for a linear process to be short-range dependent.

By Theorem 3, Propositions 3 and 4, we have

**Theorem 6.** Let \( \gamma \geq 0 \) and \( 0 \leq \mu \leq 1 \). Assume (40), \( \mathbb{E}(|\varepsilon_1|^{2+2\gamma+\mu}) < \infty \) and
\[
\int_{\mathbb{R}} |f_\varepsilon'(u)|^2 w_{\gamma+\mu}(du) < \infty.
\]
Then \( \{\sqrt{n}(P_n - P)g : g \in G_{\gamma,\mu}\} \) converges weakly to a tight Gaussian process.

**Corollary 2.** Let \( \gamma \geq 0 \). Assume (40), \( \mathbb{E}(|\varepsilon_1|^{2+\gamma}) < \infty \), \( \sup_u f_\varepsilon(u) < \infty \) and
\[
\int_{\mathbb{R}} |f_\varepsilon'(u)|^2 w_{\gamma+1}(du) + \int_{\mathbb{R}} |f_\varepsilon''(u)|^2 du < \infty.
\]
Then \( \{R_n(s)(1 + |s|)^{\gamma/2}, s \in \mathbb{R}\} \) converges weakly to a tight Gaussian process.

The corollary easily follows from Propositions 3, 4 and Corollary 1. We omit the details of the proof.

6.2.2 Long-memory linear processes

Let \( a_0 = 1 \) and \( a_n = n^{-\beta}\ell(n), n \geq 1 \), where \( 1/2 < \beta < 1 \) and \( \ell \) is a slowly varying function (Feller, 1971, p. 275). Then the covariances are not summable and we say that \( (X_t)_{t \in \mathbb{Z}} \) is long-range dependent or long-memory. Let \( K \) be a measurable function such that \( K_\infty(x) := \mathbb{E}[K(X_1 + x)] \) is in \( C^p \), the class of functions having up to \( p \)th order derivatives; let \( \sigma_{n,p}^2 = n^{2-\nu(2\beta-1)}\ell^p(n) \) and
\[
S_n(K;p) = \sum_{i=1}^{n} \left[ K(X_i) - \sum_{j=0}^{p} K^{(j)}(0)U_{i,j} \right], \text{ where } U_{n,r} = \sum_{0 \leq j_1 < \cdots < j_r} \prod_{s=1}^{r} a_{j_s} \varepsilon_{n-j_s}. \quad (42)
\]
Ho and Hsing (1996, 1997) initiated the study of such expansions, which are closely related to chaotic representations, Volterra processes (Doukhan, 2003) and Hermite expansions if
$X_n$ is Gaussian. Here we consider the weak convergence of $\{S_n(K; p) : K \in \mathcal{K}_\gamma\}$ for the class $\mathcal{K}_\gamma = \mathcal{G}^*_{\gamma, 0, 0}$; recall (21) for the definition of $\mathcal{G}^*_{\gamma, \mu, 0}$. If $(1 + p)(2\beta - 1) < 1$, then the limiting distribution of $\{\sigma_{n,p+1} S_n(K; p) : K \in \mathcal{K}_\gamma\}$ is degenerate since it forms a line of multiples of the multiple Wiener-Ito integral $Z_{p+1, \beta}$ (cf. Corollary 3 in Wu (2003a)). See Major (1981) for the definition of multiple Wiener-Ito integrals. To have a complete characterization, one needs to consider the case in which $(1 + p)(2\beta - 1) > 1$. It turns out that, with the help of Proposition 1, we are able to show that the limiting distribution is a non-degenerate Gaussian process with a $\sqrt{n}$-normalization.

Recall $Y_{n-1} = X_n - \varepsilon_n$. Let $S_n^0(y; p) = \sum_{i=1}^n L_{i-p}^0(\mathcal{F}, y)$, where

$$L_{i-p}^0(\mathcal{F}, y) = F_\varepsilon(y - Y_{n-1}) - F(y) - \sum_{i=1}^p (-1)^i F^{(i)}(y) U_{n,i}.$$  \hfill (43)

Let $L_p(\mathcal{F}, y) = [1_{X_n \leq y} - F_\varepsilon(y - Y_{n-1})] + L_{p}^0(\mathcal{F}, y)$, $l_p^0(\mathcal{F}, y) = \partial L_{p}^0(\mathcal{F}, y)/\partial y$, $s_n^0(y; p) = \sum_{i=1}^n l_p^0(\mathcal{F}, y)$ and $S_n(y; p) = \sum_{i=1}^n L_p(\mathcal{F}, y)$.

**Theorem 7.** Assume $\mathbb{E}(|\varepsilon_1|^{4+\gamma}) < \infty$ for some $\gamma \geq 0$, $f_\varepsilon \in \mathcal{C}^{p+1}$ and

$$\sum_{r=0}^{p+1} \int \mathbb{R} |f_\varepsilon^{(r)}(x)|^2 w_\gamma(dx) < \infty.$$  \hfill (44)

(i) If $p < (2\beta - 1)^{-1} - 1$, then

$$\frac{1}{\sigma_{n,p+1}} \{S_n(K; p) : K \in \mathcal{K}_\gamma\} \Rightarrow \{K^{(p+1)}(0) : K \in \mathcal{K}_\gamma\} Z_{p+1, \beta}.$$  \hfill (45)

(ii) If $p > (2\beta - 1)^{-1} - 1$, then $\{n^{-1/2} S_n(K; p) : K \in \mathcal{K}_\gamma\}$ converges weakly to a tight Gaussian process.

**Proof.** As in Wu (2003a), let $\theta_{n,p} = |a_{n-1}| [a_{n-1}| + A_n^{1/2}(4) + A_n^{p/2}(2)]$ and $\Theta_{n,p} = \sum_{k=1}^n \theta_{k,p}$, where $A_n(k) = \sum_{i=n}^\infty |a_i|^k$. By Karamata’s theorem (Feller, 1971, p. 281), $A_n(k) = \mathcal{O}(n|a_n|^k)$, $k \geq 2$. (i) It follows from Theorem 1 and Corollary 3 in Wu (2003a). (ii) Since $(1 + p)(2\beta - 1) > 1$,

$$\sum_{n=1}^\infty \theta_{n,p} = \sum_{n=1}^\infty \mathcal{O}[a_n^2 + |a_n|(na_n^4)^{1/2} + |a_n|(na_n^2)^{p/2}] < \infty.$$  \hfill (46)
By Lemma 9 in Wu (2003a), the condition \( \sum_{r=0}^{p} \int_{\mathbb{R}} |f^{(r)}(x)|^2 w_\gamma(dx) < \infty \) together with \( \mathbb{E}(|\varepsilon_1|^{4+\gamma}) < \infty \) implies that
\[
\int_{\mathbb{R}} \| P_1 L(F_n, t) \|^2 w_\gamma(dt) = \mathcal{O}(\theta_{n,p}^2). \tag{47}
\]

Using the same argument therein, it can be shown that (44) entails the similar result
\[
\int_{\mathbb{R}} \| P_1 t^0(F_n, t) \|^2 w_\gamma(dt) = \mathcal{O}(\theta_{n,p}^2). \tag{48}
\]

By Lemma 8 in Wu (2003a), \( \mathbb{E}(|\varepsilon_1|^{\max(1+\gamma, 2)}) < \infty \) and (44) imply that \( S_n(K; p) \) has the representation
\[
S_n(K; p) = -\int_{\mathbb{R}} K'(x) S_n(x; p) dx = -\sum_{i=1}^{n} \int_{\mathbb{R}} K'(x) L_p(F_i, x) dx. \tag{49}
\]

Combining (47) and (46), since \( K \in \mathcal{K}_\gamma \),
\[
\left\| P_0 \int_{\mathbb{R}} K'(y) L(F_n, y) dy \right\| \leq \sqrt{\int_{\mathbb{R}} [K'(y)]^2 w_{-\gamma}(dy)} \int_{\mathbb{R}} \| P_0 L(F_n, t) \|^2 w_\gamma(dt) = \mathcal{O}(\theta_{n+1,p})
\]
are summable and by Lemma 1 the finite-dimensional convergence follows.

We now use the truncation argument as in the proofs of Propositions 1 and 2 to establish the tightness. Since \( P_0 L(F_n, t) = P_0 L^0(F_n, t) \) for \( n \neq 0 \), (47) implies that \( \int_{\mathbb{R}} \| P_1 L^0(F_n, t) \|^2 w_\gamma(dt) = \mathcal{O}(\theta_{n,p}^2) \). By Lemma 4 and (46), we have
\[
\lim_{r \to \infty} \limsup_{n \to \infty} \frac{1}{n} \int_{|x|>r} \| S_n^0(x; p) \|^2 w_\gamma(dx) = 0, \tag{50}
\]
and
\[
\limsup_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}} \| s_n^0(x; p) \|^2 w_\gamma(dx) < \infty. \tag{51}
\]

Applying the maximal inequality (53) of Lemma 3 with \( \mu = 0 \),
\[
\frac{1}{n} \mathbb{E} \left[ \sup_{x} |S_n^0(x; p)|^2 (1 + |x|)^\gamma \right] \leq \frac{1}{n} \int_{\mathbb{R}} [\| S_n^0(x; p) \|^2 + \| s_n^0(x; p) \|^2] w_\gamma(dx) = \mathcal{O}(1). \tag{52}
\]

Note that \( S_n(x; p) = n[F_n(x) - \tilde{F}_n(x)] + S_n^0(x; p) \). By (49),
\[
S_n(K; p) = -\int_{\mathbb{R}} K'(x) n[F_n(x) - F(x)] dx - \int_{\mathbb{R}} K'(x) S_n^0(x; p) dx
\]
\[ = \sqrt{n} M_n(K) - \int_{|x| > r} K'(x) S_n^0(x; p) dx - \int_{-r}^r K'(x) S_n^0(x; p) dx. \]

Since \( \mathbb{E}(|X_1|^{\gamma+1}) < \infty \), by (i) of Proposition 1, the process \( \{M_n(K) \colon K \in \mathcal{K}_\gamma\} \) is tight.

For the second term, by the Cauchy-Schwarz inequality,
\[
\frac{1}{n} \left\| \int_{|x| > r} K'(x) S_n^0(x; p) dx \right\|^2 \leq \int_{|x| > r} [K'(x)]^2 w_{-\gamma}(dx) \times \frac{1}{n} \int_{|x| > r} \|S_n^0(x; p)\|^2 w_{\gamma}(dx),
\]
which converges to 0 by first letting \( n \to \infty \) and then \( r \to \infty \). Now we deal with the last term. Using integration by parts,
\[
- \int_{-r}^r K'(x) S_n^0(x; p) dx = \int_{-r}^r K(x) s_n^0(x; p) dx - K(r) S_n^0(r; p) + K(-r) S_n^0(-r; p)
\]
As in the proof of Proposition 2, (51) implies that \( \{n^{-1/2} \int_r^r K(x) s_n^0(x; p) dx, K \in \mathcal{K}_\gamma\} \) is tight. By (52), \( \|S_n^0(\pm r; p)\| = O(\sqrt{n}) \). Since \( |K(x)|^2 \leq C_{\gamma}(1 + |x|)^{\gamma+1} \), it is easily seen that \( \{n^{-1/2} K(\pm r) S_n^0(\pm r; p), K \in \mathcal{K}_\gamma\} \) is also tight.

\[ \diamond \]

It is unclear how to generalize Theorem 7 to long-range dependent heavy-tailed processes, linear fields (non-causal) and other long-range dependent processes. The special case \( p = 1 \) is considered in many earlier results; see Giraitis and Surgailis (1999), Doukhan et al (2002) and Doukhan et al (2004). Limit theorems for heavy-tailed processes are presented in Hsing (1999), Koul and Surgailis (2001), Surgailis (2002), Wu (2003b) and Pipiras and Taqqu (2003). It seems that there is no easy way to establish \( p \)th order expansions for \( p \geq 2 \).

## 7 Inequalities

The inequalities presented in this section are of independent interest and they may have wider applicability. They are used in the proofs of the results in other sections.

**Lemma 3.** Let \( H \in \mathcal{AC} \). (i) Let \( \mu \leq 1 \) and \( \gamma \in \mathbb{R} \). Then there exists \( C_{\gamma,\mu} < \infty \) such that
\[
\sup_{x \in \mathbb{R}} [H^2(x)(1 + |x|)^\gamma] \leq C_{\gamma,\mu} \int_{\mathbb{R}} H^2(u) w_{\gamma-\mu}(du) + C_{\gamma,\mu} \int_{\mathbb{R}} [H'(u)]^2 w_{\gamma+\mu}(du). \quad (53)
\]
(ii) Let \( \gamma > 0, \mu = 1 \) and \( H(0) = 0 \). Then
\[
\sup_{x \in \mathbb{R}} [H^2(x)(1 + |x|)^{-\gamma}] \leq \frac{1}{\gamma} \int_{\mathbb{R}} [H'(u)]^2 w_{-\gamma+1}(du) \quad (54)
\]
and
\[ \int_{\mathbb{R}} H^2(u) w_{-\gamma-1}(du) \leq \frac{4}{\gamma^2} \int_{\mathbb{R}} [H'(u)]^2 w_{-\gamma+1}(du). \] (55)

(iii) Let \( \gamma > 0 \) and \( H(\pm \infty) = 0 \). Then
\[ \sup_{x \in \mathbb{R}} [H^2(x)(1 + |x|)^\gamma] \leq \frac{1}{\gamma} \int_{\mathbb{R}} [H'(u)]^2 w_{\gamma+1}(du) \] (56)
and
\[ \int_{\mathbb{R}} H^2(u) w_{\gamma-1}(du) \leq \frac{4}{\gamma^2} \int_{\mathbb{R}} [H'(u)]^2 w_{\gamma+1}(du). \] (57)

**Proof.** (i) By Lemma 4 in Wu (2003a), for \( t \in \mathbb{R} \) and \( \delta > 0 \) we have
\[ \sup_{t \leq s \leq t + \delta} H^2(s) \leq \frac{2}{\delta} \int_{t}^{t+\delta} H^2(u)du + 2\delta \int_{t}^{t+\delta} [H'(u)]^2 du. \] (58)

We first consider the case \( \mu < 1 \). Let \( \alpha = \frac{1}{1 - \mu} \). In (58) let \( t = t_n = n^a \) and \( \delta = \delta_n = (n + 1)^{\alpha} - n^\alpha, n \in \mathbb{N} \) and \( I_n = [t_n, t_{n+1}] \). Since \( \delta_n \sim \alpha n^{\alpha-1} \) as \( n \to \infty \),
\[ \sup_{x \in I_n} [H^2(x)(1 + x)^\gamma] \leq 2 \sup_{x \in I_n} (1 + x)^\gamma \left[ \delta_n^{-1} \int_{I_n} H^2(u)du + \delta_n \int_{I_n} [H'(u)]^2 du \right] \leq C \int_{I_n} H^2(u) w_{\gamma-\mu}(du) + C \int_{I_n} [H'(u)]^2 w_{\gamma+\mu}(du). \] (59)

It is easily seen in view of (58) that (59) also holds for \( n = 0 \) by choosing a suitable \( C \). By summing (59) over \( n = 0, 1, \ldots \), we obtain (53) with \( \sup_{x \in \mathbb{R}} \) replaced by \( \sup_{x \geq 0} \). The other side \( x \leq 0 \) similarly follows.

If \( \mu = 1 \), we let \( t_n = 2^n, \delta_n = t_{n+1} - t_n = t_n \) and \( I_n = [t_n, t_{n+1}], n = 0, 1, \ldots \) The argument above similarly yields the desired inequality.

(ii) Let \( s \geq 0 \). Since \( H(s) = \int_{0}^{s} H'(u)du \), by the Cauchy-Schwarz inequality, (54) follows from
\[ H^2(s) \leq \int_{0}^{s} |H'(u)|^2 (1 + u)^{1-\gamma} du \times \int_{0}^{s} (1 + u)^{\gamma-1} du \]
\[ \leq \int_{\mathbb{R}} [H'(u)]^2 w_{-\gamma+1}(du) \times \frac{(1 + s)^\gamma - 1}{\gamma}. \]

Applying Theorem 1.14 in Opic and Kufner (1990, p. 13) with \( p = q = 2 \), the Hardy-type inequality (55) easily follows. The proof of (iii) is similar as (ii). \( \Diamond \)

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Lemma 4. Let $m$ be a measure on $\mathbb{R}$, $A \subset \mathbb{R}$ a measurable set and $T_n(\theta) = \sum_{i=1}^{n} h(\theta, F_i)$, where $h$ is a measurable function. Then
\[
\sqrt{\int_A \|T_n(\theta) - \mathbb{E}[T_n(\theta)]\|^2 m(d\theta)} \leq \sqrt{n} \sum_{j=0}^{\infty} \sqrt{\int_A \|P_0 h(\theta, F_j)\|^2 m(d\theta)}.
\] (60)

Proof. For $j = 0, 1, \ldots$ let $T_{n,j}(\theta) = \sum_{i=1}^{n} \mathbb{E}[h(\theta, F_i) | F_{i-j}]$ and $\lambda_j^2 = \int_A \|P_0 h(\theta, F_j)\|^2 m(d\theta)$, $\lambda_j \geq 0$. By the orthogonality of $\mathbb{E}[h(\theta, F_i) | F_{i-j}] - \mathbb{E}[h(\theta, F_i) | F_{i-j-1}]$, $i = 1, 2, \ldots, n,$
\[
\int_A \|T_{n,j}(\theta) - T_{n,j-1}(\theta)\|^2 m(d\theta) = n \int_A \|\mathbb{E}[h(\theta, F_1) | F_{1-j}] - \mathbb{E}[h(\theta, F_1) | F_{j}]\|^2 m(d\theta)
\]
\[
= n \int_A \|P_{j} h(\theta, F_{1})\|^2 m(d\theta) = n \lambda_j^2.
\]

Note that $T_n(\theta) = T_{n,0}(\theta)$. Let $\Delta = \sum_{j=0}^{\infty} \lambda_j$. By the Cauchy-Schwarz inequality,
\[
\int_A \mathbb{E}[T_n(\theta) - \mathbb{E}[T_n(\theta)]]^2 m(d\theta) = \int_A \mathbb{E} \left\{ \sum_{j=0}^{\infty} [T_{n,j}(\theta) - T_{n,j+1}(\theta)] \right\}^2 m(d\theta)
\]
\[
\leq \Delta \int_A \mathbb{E} \left\{ \sum_{j=0}^{\infty} \lambda_j^{-1} [T_{n,j}(\theta) - T_{n,j+1}(\theta)]^2 \right\} m(d\theta) = n \Delta^2
\]
and (60) follows. \hfill \Box

Lemma 5 easily follows from Burkholder’s inequality. We omit the proof.

Lemma 5. Let $(D_i)_{1 \leq i \leq n}$ be $L^q$ ($q > 1$) martingale differences. Then
\[
\|D_1 + \ldots + D_n\|_{q}^{\min(q,2)} \leq [18 q^{3/2} (q-1)^{-1/2}]^{\min(q,2)} \sum_{i=1}^{n} \|D_i\|_{q}^{\min(q,2)}.
\] (61)

Lemma 6 below gives a simple maximal inequality. Weaker and special versions of it can be found in Doob (1953), Wu and Woodroofe (2004) and Billingsley (1968). It has the advantage that the dependence structure of $\{Z_i\}$ can be arbitrary.

Lemma 6. Let $q > 1$ and $Z_i$, $1 \leq i \leq 2^d$, be random variables in $L^q$, where $d$ is a positive integer. Let $S_n = Z_1 + \ldots + Z_n$ and $S^*_n = \max_{1 \leq i \leq n} |S_i|$. Then
\[
\|S^*_n\|_q \leq \sum_{r=0}^{d} \left[ \sum_{m=1}^{2^{d-r}} \|S_{2^r,m} - S_{2^r(m-1)}\|_q^q \right]^{1/q}.
\] (62)
Proof. Let \( p = \frac{q}{(q - 1)} \) and \( \Lambda = \sum_{r=0}^{d} \lambda_r^{-p} \), where

\[
\lambda_r = \left( \sum_{m=1}^{2^{d-r}} \| S_{2^r m} - S_{2^r (m-1)} \|_q^q \right)^{-\frac{1}{p+q}}.
\]

For the positive integer \( k \leq 2^d \), write its dyadic expansion \( k = 2^{r_1} + \ldots + 2^{r_j} \), where \( 0 \leq r_j < \ldots < r_1 \leq d \), and \( k(i) = 2^{r_1} + \ldots + 2^{r_i} \). By Hölder’s inequality,

\[
|S_k|^q \leq \left[ \sum_{i=1}^{j} |S_{k(i)} - S_{k(i-1)}| \right]^q
\leq \left[ \sum_{i=1}^{j} \lambda_{r_i}^{-p} \right]^q \left[ \sum_{i=1}^{j} \lambda_{r_i}^q |S_{k(i)} - S_{k(i-1)}|^q \right]
\leq \Lambda^q/p \sum_{i=1}^{j} \lambda_{r_i}^q \sum_{m=1}^{2^{d-r_i}} |S_{2^{r_i} m} - S_{2^{r_i} (m-1)}|^q
\leq \Lambda^q/p \sum_{i=0}^{d} \lambda_{r_i}^q \sum_{m=1}^{2^{d-r}} |S_{2^r m} - S_{2^r (m-1)}|^q,
\]

which entails \( \| S_{2^d}^* \|_q^q \leq \Lambda^q/p \sum_{r=0}^{d} \lambda_{r}^q \lambda_r^{-p-q} = \Lambda^q \) and hence (62).

8 Proofs of Theorems 1 and 2

Following (5), let \( G_n(s) = n^{1/2}[F_n(x) - \tilde{F}_n(x)] \) and \( Q_n(s) = n^{1/2}[\tilde{F}_n(x) - F(x)] \). Then \( R_n(s) = G_n(s) + Q_n(s) \). Sections 8.1 and 8.2 deal with \( G_n \) and \( Q_n \) respectively. Theorems 1 and 2 are proved in Sections 8.3 and 8.4.

8.1 Analysis of \( G_n \)

The main result is this section is Lemma 9 which concerns the weak convergence of \( G_n \).

**Lemma 7.** Let \( q \geq 2 \). Then there is a constant \( C_q < \infty \) such that

\[
\| G_n(y) - G_n(x) \|_q^q \leq C_q n^\max(1,q/4)-q/2 |F(y) - F(x)|
+ C_q(y-x)^{q/2-1} \int_x^y \mathbb{E}[f_{x}^{2/2}(u|\mathcal{F}_0)]du
\]  

(63)
\[ ||G_n(x)||_q^q \leq C_q \min[F(x), 1 - F(x)]. \quad (64) \]

**Proof.** Let \( d_i(s) = 1_{X_i \leq s} - \mathbb{E}(1_{X_i \leq s}|\mathcal{F}_{i-1}) \), \( d_i = d_i(y) - d_i(x) \) and \( D_i = d_i^2 - \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \). Wu (2003a) deals with the special case in which \( q = 4 \) [cf. Inequality (48) therein]. Let \( q' = q/2 \). By Burkholder’s inequality (Chow and Teicher, 1978),
\[
||G_n(y) - G_n(x)||_q^q = n^{-q/2} \mathbb{E}(|d_1 + \ldots + d_n|^q) \\
\leq C_q n^{-q/2} \mathbb{E}[(|d_1^2 + \ldots + d_n^2|)^{q/2}] \\
\leq C_q n^{-q/2} \left\| \sum_{i=1}^n D_i \right\|_{q'}^{q'} + C_q n^{-q/2} \mathbb{E} \left\{ \left[ \sum_{i=1}^n \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \right]^{q/2} \right\}. \quad (65)
\]

Since \( D_i, i \in \mathbb{Z} \), form stationary martingale differences, by Lemma 5 we have
\[
\left\| \sum_{i=1}^n D_i \right\|_{q'} \leq C_q n^{\max(1,q'/2)} ||D_1||_{q'}^{q'} \\
\leq C_q n^{\max(1,q'/2)} 2^{q'-1} \left[ ||d_1^2||_{q'}^{q'} + \mathbb{E}(d_1^2|\mathcal{F}_0)||_{q'}^{q'} \right] \\
\leq C_q n^{\max(1,q'/2)} 2^{q'} ||d_1||_{q'}^{q'}. \quad (66)
\]

where we have applied Jensen’s inequality \( \mathbb{E}(d_1^2|\mathcal{F}_0)||_{q'}^{q'} \leq ||d_1||_{q'}^{q'} \). Notice that \( |d_1| \leq 1 \),
\[
||d_1^2||_{q'}^{q'} \leq ||d_1||_{q'}^{q'} \leq 2^{q'-1} ||1_{x \leq X_i \leq y}||_{q'}^{q'} + \mathbb{E}(1_{x \leq X_i \leq y}|\mathcal{F}_0)||_{q'}^{q'} \leq 2^{q'} [F(y) - F(x)]. \quad (67)
\]

On the other hand, since \( q' \geq 1 \) and \( \mathbb{E}(d_1^2|\mathcal{F}_0) \leq \mathbb{E}(1_{x \leq X_i \leq y}|\mathcal{F}_0) \), we have by Hölder’s inequality with \( p' = q'/2 \) that
\[
\mathbb{E} \left\{ \left[ \sum_{i=1}^n \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \right]^{q/2} \right\} \leq n^{q'} \mathbb{E}(d_1^2|\mathcal{F}_0)||_{q'}^{q'} \\
\leq n^{q'} \mathbb{E} \left\{ \left[ \int_x^y f_{\varepsilon}(u|\mathcal{F}_0)du \right]^{q'} \right\} \\
\leq n^{q'} \mathbb{E} \left[ (y-x)^{q'/2} \int_x^y f_{\varepsilon}'(u|\mathcal{F}_0)du \right]. \quad (68)
\]

Combining (65), (66), (67) and (68), we have (63).

To show (64), we let in (65) that \( d_i = d_i(x) = 1_{X_i \leq x} - \mathbb{E}(1_{X_i \leq x}|\mathcal{F}_{i-1}) \). Then
\[
\mathbb{E}(|d_1 + \ldots + d_n|^q) \leq C_q n^{\max(1,q/2)} ||d_1||_{q'}^{q'} \\
\leq C_q n^{q/2} ||d_1||^2 \leq C_q n^{q/2} F(x)[1 - F(x)]
\]

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Lemma 8. Let \( q > 2 \). Then there exists a constant \( C_q < \infty \) such that
\[
\mathbb{E} \left[ \sup_{0 \leq s < b} |G_n(a + s) - G_n(a)|^q \right] \leq C_q d^q n^{\max(1,q/4)-q/2}[F(a+b) - F(a)]
\]
\[+ C_q b^{q/2-1}[1 + n^{q/2}2^{d(1-q/2)}] \int_a^{a+b} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du. \tag{69}\]
holds for all \( b > 0 \), \( a \in \mathbb{R} \) and \( n, d \in \mathbb{N} \). In particular, for \( d = 1 + [(\log n)/((1-2/q)\log 2)] \), we have
\[
\mathbb{E} \left[ \sup_{0 \leq s < b} |G_n(a + s) - G_n(a)|^q \right] \leq C_q (\log n)^q n^{\max(1,q/4)-q/2}[F(a+b) - F(a)]
\]
\[+ C_q b^{q/2-1} \int_a^{a+b} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du. \tag{70}\]

Proof. Let \( h = b2^{-d}, Z_j = G_n(a+jh) - G_n(a+(j-1)h), j = 1, \ldots, 2^d \) and \( S_j = Z_1 + \ldots + Z_j \).

By Lemma 7,
\[
\|S_{2^r m} - S_{2^r (m-1)}\|^q \leq C_q n^{\max(1,q/4)-q/2}[F(a+2^r mh) - F(a+2^r (m-1)h)]
\]
\[+ C_q (2^r h)^{q/2-1} \int_{a+2^r (m-1)h}^{a+2^r mh} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du\]

Hence
\[
\sum_{m=1}^{2^d - r} \|S_{2^r m} - S_{2^r (m-1)}\|^q \leq C_q n^{\max(1,q/4)-q/2}[F(a+b) - F(a)]
\]
\[+ C_q (2^r h)^{q/2-1} \int_a^{a+b} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du. \tag{71}\]

By Lemma 6,
\[
\|S_{2^d}^*\|_q \leq \sum_{r=0}^{d} \left\{ C_q n^{\max(1,q/4)-q/2}[F(a+b) - F(a)] \right\}^{1/q}
\]
\[+ \sum_{r=0}^{d} \left\{ C_q (2^r h)^{q/2-1} \int_a^{a+b} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du \right\}^{1/q}\]
\[\leq d \left\{ C_q n^{\max(1,q/4)-q/2}[F(a+b) - F(a)] \right\}^{1/q}
\]
\[+ \left\{ C_q (2^d h)^{q/2-1} \int_a^{a+b} \mathbb{E}[f_{\xi}^{q/2}(u|\mathcal{F}_0)]du \right\}^{1/q}. \]
Recall $\tilde{F}_n(x) = n^{-1} \sum_{i=1}^n F_\varepsilon(x|\mathcal{F}_{i-1})$. Let $B_j = \sqrt{n}[\tilde{F}_n(a + jh) - \tilde{F}_n(a + (j-1)h)]$, $j = 1, \ldots, 2^d$, and $q' = q/2$. Since $0 \leq F_\varepsilon \leq 1$, by Hölder’s inequality,

$$
\|B_j\|_{q'}^q = n^{-q'} \mathbb{E} \left[ \sum_{i=1}^n \int_{a+(j-1)h}^{a+jh} f_\varepsilon(u|\mathcal{F}_{i-1}) \, du \right]^q
\leq n^{-q'} n^{q-1} \sum_{i=1}^n \mathbb{E} \left[ \int_{a+(j-1)h}^{a+jh} f_\varepsilon(u|\mathcal{F}_{i-1}) \, du \right]^q
\leq n^{-q'} n^{q-1} \sum_{i=1}^n \mathbb{E} \left[ \int_{a+(j-1)h}^{a+jh} f_\varepsilon(u|\mathcal{F}_{i-1}) \, du \right]^q
\leq n'^q h'^q \int_{a+(j-1)h}^{a+jh} \mathbb{E}[f_\varepsilon^{q/2}(u|\mathcal{F}_0)] \, du.
$$

Therefore,

$$
\mathbb{E} \left[ \max_{j \leq 2^d} B_j^q \right] \leq \mathbb{E} \left[ \sum_{j=1}^{2^d} B_j^q \right] \leq n'^q h'^q \int_a^{a+b} \mathbb{E}[f_\varepsilon^{q/2}(u|\mathcal{F}_0)] \, du. \tag{72}
$$

Observe that

$$
G_n(a + h[s/h]) - \max_{j \leq 2^d} B_j \leq G_n(a + s) \leq G_n(a + h[s/h + 1]) + \max_{j \leq 2^d} B_j.
$$

Hence (69) follows from (71), (72) and

$$
\sup_{0 \leq s \leq b} |G_n(a + s) - G_n(a)| \leq \sup_{0 \leq s \leq b} |G_n(a + h[s/h + 1]) - G_n(a)| \\
+ \sup_{0 \leq s \leq b} |G_n(a + h[s/h]) - G_n(a)| + 2 \max_{j \leq 2^d} B_j
\leq 2 \max_{j \leq 2^d} |G_n(a + hj) - G_n(a)| + 2 \max_{j \leq 2^d} B_j
= 2S_{2^d} + 2 \max_{j \leq 2^d} B_j
$$

by noticing that $h = 2^{-d}b$.

For $d = 1 + \lfloor (\log n)/((1 - 2/q) \log 2) \rfloor$, we have $n^{q/2}2^{d(1-q/2)} \leq 1$ and hence (70) is an easy consequence of (69).

\[\diamondsuit\]

**Lemma 9.** Let $\gamma \geq 0$ and $q > 2$. Assume $\mathbb{E}[|X_1|^{\gamma} + \log(1 + |X_1|)] < \infty$ and (7). Then (i)

$$
\mathbb{E} \left[ \sup_{s \in \mathbb{R}} |G_n(s)|^q (1 + |s|)^\gamma \right] = \mathcal{O}(1) \tag{73}
$$
and (ii) the process \( \{G_n(s)(1 + |s|)^{\gamma/q}, s \in \mathbb{R}\} \) is tight and it converges to a tight Gaussian process.

**Remark 6.** In Lemma 9, the logarithm term \( \log(1 + |X_1|) \) is not needed if \( \gamma > 0 \). ◦

**Proof.** (i) Without loss of generality we show that \( \mathbb{E}[\sup_{s \geq 0} |G_n(s)|^q (1 + |s|)^\gamma] = \mathcal{O}(1) \) since the case when \( s < 0 \) follows similarly. Let \( \alpha_n = (\log n)^q n^{\max(1, q/4) - q/2} \). By (7) and (70) of Lemma 8 with \( a = b = 2^k \),

\[
\sum_{k=1}^{\infty} (1 + 2^k)^\gamma \mathbb{E} \left[ \sup_{2^k \leq s < 2^{k+1}} |G_n(s) - G_n(2^k)|^q \right] \\
\leq C_q \sum_{k=1}^{\infty} (1 + 2^k)^\gamma \alpha_n [F(2^{k+1}) - F(2^k)] \\
+ C_q \sum_{k=1}^{\infty} (1 + 2^k)^\gamma (2^k)^{q/2-1} \int_{2^k}^{2^{k+1}} \mathbb{E}[f_{\gamma/2}^q(u|\mathcal{F}_0)] du \\
\leq C_{\gamma,q} \alpha_n \int_2^\infty f(u)(1 + u)^\gamma du + C_{\gamma,q} \int_2^\infty (1 + u)^\gamma u^{q/2-1} \mathbb{E}[f_{\gamma/2}^q(u|\mathcal{F}_0)] du \\
\leq C_{\gamma,q} \alpha_n + C_{\gamma,q} = \mathcal{O}(1). \quad (74)
\]

Let \( A_\gamma(d) = \sum_{k=1}^{d} (1 + 2k)^\gamma \). Then \( A_\gamma([\log_2 v] + 1) \leq C(1 + v)^\gamma \) if \( \gamma > 0 \) and \( A_\gamma([\log_2 v] + 1) \leq \log_2 v \) if \( \gamma = 0 \). By (64) of Lemma 7, we have

\[
\sum_{k=1}^{\infty} (1 + 2^k)^\gamma \|G_n(2^k)\|_q^q \leq \sum_{k=1}^{\infty} (1 + 2^k)^\gamma C_q \int_{2^k}^\infty f(v) dv \\
\leq C_q \int_2^\infty f(v) A_\gamma([\log_2 v] + 1) dv < \infty. \quad (75)
\]

Thus (73) follows from (74) and (75).

(ii) It is easily seen that the argument in (i) entails

\[
\lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{E} \left[ \sup_{|s| > r} |G_n(s)|^q (1 + |s|)^\gamma \right] = 0. \quad (76)
\]

For \( \delta \in (0, 1) \) let the interval \( I_k = I_k(\delta) = [k\delta, (k + 1)\delta] \). Then

\[
\sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |G_n(s)(1 + |s|)^{\gamma/q} - G_n(t)(1 + |t|)^{\gamma/q}| \\
\leq \sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |(1 + |s|)^{\gamma/q}[G_n(s) - G_n(t)]|
\]
\[ + \sup_{s,t \in [-r,r], 0 \leq s - t \leq \delta} |G_n(t)[(1 + |s|)^{\gamma/q} - (1 + |t|)^{\gamma/q}]| \leq (1 + r)^{\gamma/q} \sup_{s,t \in [-r,r], 0 \leq s - t \leq \delta} |G_n(s) - G_n(t)| + C_{r,\gamma,q} \delta \sup_{u \in [-r,r]} |G_n(u)| \quad (77) \]

By (i), \( \| \sup_{u \in \mathbb{R}} |G_n(u)| \| = \mathcal{O}(1) \). On the other hand, by Lemma 8,

\[
\epsilon^{-q} \sum_{k = -[r/\delta] - 1}^{[r/\delta] + 1} \left\{ C_{q,\alpha_n} \mathbb{E}[X_1 \in I_k] + C_{q,\delta^{q/2 - 1}} \int_{I_k} \mathbb{E}\left[f^{q/2}_\varepsilon(u|\mathcal{F}_0)\right] du \right\} 
\leq \epsilon^{-q} C_{q,\alpha_n} + \epsilon^{-q} C_{q,\delta^{q/2 - 1}} \int_{\mathbb{R}} \mathbb{E}\left[f^{q/2}_\varepsilon(u|\mathcal{F}_0)\right] du. 
\]

By (7), \( \int_{\mathbb{R}} \mathbb{E}\left[f^{q/2}_\varepsilon(u|\mathcal{F}_0)\right] du < \infty \). Hence

\[
\limsup_{n \to \infty} \mathbb{P}\left[ \sup_{s,t \in [-r,r], 0 \leq s - t \leq \delta} |G_n(s) - G_n(t)| > 2\epsilon \right] \leq \epsilon^{-q} C_{q,\delta^{q/2 - 1}},
\]

which implies the tightness of \( \{G_n(s), -r \leq s \leq r\} \) for fixed \( r \). So (ii) follows from (76) and (77).

\[ \diamond \]

8.2 Analysis of \( Q_n \)

It is relatively easier to handle \( Q_n \) since it is a differentiable function. The Hardy-type inequalities (cf Lemma 3) are applicable.

**Lemma 10.** Assume (12). Then (i)

\[
\mathbb{E}\left[ \sup_{s \in \mathbb{R}} |Q_n(s)|^2 (1 + |s|)^{2\gamma/q} \right] = \mathcal{O}(1)
\quad (78)
\]

and (ii) the process \( \{Q_n(s)(1 + |s|)^{\gamma/q}, s \in \mathbb{R}\} \) is tight.

**Proof.** Let \( \gamma' = 2\gamma/q \). (i) By (56) of Lemma 3,

\[
\sup_{|s| \geq r} [Q_n^2(s)(1 + |s|)^{\gamma'}] \leq \frac{1}{\gamma'} \int_{|s| \geq r} [Q_n'(s)]^2 w_{1+\gamma'}(ds). 
\]
By Lemma 4,
\[
\left\| \sup_{|s| \geq r} |Q_n(s)|(1 + |s|)^{\gamma'/2} \right\| \leq \frac{1}{\sqrt{\gamma'}} \sum_{j=0}^{\infty} \sqrt{\int_{|s| \geq r} \|P_0 f_{\epsilon}(\theta|\mathcal{F}_j)\|^2 w_{\gamma'+1}(d\theta)}.
\] (79)

So (78) follows by letting \( r = 0 \) in (79).

(ii) The tightness follows from the similar argument as (ii) of Lemma 9. Let \( 0 < \delta < 1 \). Then
\[
\sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |Q_n(s)(1 + |s|)^{\gamma'/q} - Q_n(t)(1 + |t|)^{\gamma'/q}|
\leq \sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |(1 + |s|)^{\gamma'/q}[Q_n(s) - Q_n(t)]|
+ \sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |Q_n(t)(1 + |s|)^{\gamma'/q} - (1 + |t|)^{\gamma'/q}|
\leq C_{r,\gamma,q} \delta \sup_{u \in [-r,r]} |Q_n(u)| + C_{r,\gamma,q} \delta \sup_{u \in [-r,r]} |Q_n(u)|
\]

Notice that \( \sup_{u \in \mathbb{R}} |Q_n(u)| = O(1) \). By (53) of Lemma 3 and Lemma 4,
\[
\mathbb{E} \left[ \sup_{s \in \mathbb{R}} |Q_n'(s)|^2 \right] \leq C_{\gamma'} \int_{\mathbb{R}} \|Q_n'(s)\|^2 w_{1+\gamma'}(ds) + C_{\gamma'} \int_{\mathbb{R}} \|Q_n'(s)\|^2 w_{1-\gamma'}(ds)
\leq C_{\gamma'} \sigma^2(f_{\epsilon}, w_{1+\gamma'}) + C_{\gamma'} \sigma^2(f_{\epsilon}', w_{1-\gamma'}) = O(1).
\]

Then there exists \( C_1 < \infty \) such that for all \( n \in \mathbb{N} \),
\[
\mathbb{E} \left[ \sup_{s,t \in [-r,r], \ 0 \leq s-t \leq \delta} |Q_n(s)(1 + |s|)^{\gamma'/q} - Q_n(t)(1 + |t|)^{\gamma'/q}|^2 \right] \leq \delta^2 C_1.
\]

Notice that the upper bound in (79) goes to 0 as \( r \to \infty \). Hence (ii) obtains. \( \diamond \)

### 8.3 Proof of Theorem 1.

Observe that \( (\partial/\partial \theta) P_0 F_{\epsilon}(\theta|\mathcal{F}_j) = P_0 f_{\epsilon}(\theta|\mathcal{F}_j) \) and \( P_0 F_{\epsilon}(\theta|\mathcal{F}_j) = 0 \) when \( \theta = \pm \infty \). By (56) of Lemma 3,
\[
\sup_{\theta \in \mathbb{R}} \|P_0 F_{\epsilon}(\theta|\mathcal{F}_j)\|^2 (1 + |\theta|)^{2\gamma'/q} \leq \frac{q}{2\gamma} \int_{\mathbb{R}} |P_0 f_{\epsilon}(\theta|\mathcal{F}_j)|^2 w_{1+2\gamma'/q} d\theta.
\]

Hence by (8),
\[
\sum_{i=0}^{\infty} \sup_{\theta \in \mathbb{R}} \|P_0 F_{\epsilon}(\theta|\mathcal{F}_j)\| \leq \sqrt{q/(2\gamma)} \sigma(f_{\epsilon}, w_{1+2\gamma'/q}) < \infty,
\]

which by Lemma 1 entails the finite-dimensional convergence. Since \( R_n(s) = G_n(s) + Q_n(s) \),
the tightness and (9) follows from Lemmas 9 and 10. \( \diamond \)
8.4 Proof of Theorem 2.

Note that \( (\log n)^q n^{\max(1,q/4)-q/2} = (\log n)^q n^{1-q/2} = O(\delta_n^{q/2-1}) \). By (70) of Lemma 8, under the proposed condition we have uniformly in \( a \) that

\[
\mathbb{E} \left[ \sup_{0 \leq s < \delta_n} |G_n(a + s) - G_n(a)|^q \right] \leq C_q (\log^q n) n^{\max(1,q/4)-q/2} [F(a + \delta_n) - F(a)]
\]

\[
+ \frac{C_q \delta_n^{q/2-1} \tau^{q/2-1}}{2} \int_a^{a+\delta_n} f(u)du
\]

\[
\leq C \delta_n^{q/2-1} [F(a + \delta_n) - F(a)].
\]

Here the constant \( C \) only depends on \( \tau, \gamma, q \) and \( \mathbb{E}(|X_1|^\gamma) \). Hence

\[
\sum_{k \in \mathbb{Z}} (1 + |k\delta_n|)^\gamma \mathbb{E} \left[ \sup_{0 \leq s < \delta_n} |G_n(k\delta_n + s) - G_n(k\delta_n)|^q \right]
\]

\[
\leq \sum_{k \in \mathbb{Z}} (1 + |k\delta_n|)^\gamma C \delta_n^{q/2-1} [F(k\delta_n + \delta_n) - F(k\delta_n)] \leq C \delta_n^{q/2-1} \mathbb{E}[(1 + |X_1|^\gamma)].
\]

Let \( I_k(\delta) = [k\delta, (1 + k)\delta] \). Since \( 0 < \delta < 1 \), we have

\[
\frac{1}{2} \leq \frac{1}{1 + \delta} \leq \frac{1 + |t|}{1 + |k\delta|^\gamma} \leq \sup_{t \in I_k(\delta)} \frac{1 + |t|}{1 + |k\delta|} \leq 1 + \delta \leq 2
\]

and

\[
\sup_{t \in I_k(\delta), 0 \leq s < \delta_n} |G_n(t + s) - G_n(t)| \leq \sup_{t \in I_k(\delta), 0 \leq s < \delta_n} |G_n(t + s) - G_n(k\delta_n)|
\]

\[
+ \sup_{t \in I_k(\delta), 0 \leq s < \delta_n} |G_n(k\delta_n) - G_n(t)|
\]

\[
\leq \sup_{0 \leq u < 2\delta_n} |G_n(k\delta_n + u) - G_n(k\delta_n)|
\]

\[
+ \sup_{0 \leq s < \delta_n} |G_n(k\delta_n + s) - G_n(k\delta_n)|
\]

\[
\leq 2 \sup_{0 \leq u < 2\delta_n} |G_n(k\delta_n + u) - G_n(k\delta_n)|
\]

Therefore,

\[
\mathbb{E} \left[ \sup_{t \in \mathbb{R}} (1 + |t|)^\gamma \sup_{0 \leq s < \delta_n} |G_n(t + s) - G_n(t)|^q \right]
\]

\[
\leq \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \sup_{t \in I_k(\delta_n)} (1 + |t|)^\gamma \sup_{0 \leq s < \delta_n} |G_n(t + s) - G_n(t)|^q \right]
\]

\[
\leq C \sum_{k \in \mathbb{Z}} (1 + |k\delta_n|)^\gamma \mathbb{E} \left[ \sup_{0 \leq s < 2\delta_n} |G_n(k\delta_n + s) - G_n(k\delta_n)|^q \right] \leq C \delta_n^{q/2-1}.
\]
Note that \( R_n(s) = G_n(s) + Q_n(s) \). Then (13) follows if it holds with \( R_n \) replaced by \( G_n \) and \( Q_n \) respectively. The former is an easy consequence of the preceding inequality and Jensen’s inequality. To show that (13) holds with \( R_n \) replaced by \( Q_n \), let \( \gamma' = 2\gamma/q \). By (53) of Lemma 3 and Lemma 4, 
\[
\mathbb{E} \left[ \sup_{x \in \mathbb{R}} (1 + |x|)^{\gamma'} |Q_n'(x)|^2 \right] \leq C \int_{\mathbb{R}} \|Q_n(x)\|^2 w_{\gamma' - \mu}(dx) + C \int_{\mathbb{R}} \|Q_n'(x)\|^2 w_{\gamma' + \mu}(dx)
\]
\[
\leq C \sigma^2 (f_\varepsilon, w_{\gamma' - \mu}) + C \sigma^2 (f'_\varepsilon, w_{\gamma' + \mu}) < \infty,
\]
which entails that 
\[
\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} (1 + |t|)^{\gamma'} \sup_{|s| \leq \delta_n} |Q_n(t + s) - Q_n(t)|^2 \right\}
\]
\[
\leq \delta_n^2 \mathbb{E} \left\{ \sup_{t \in \mathbb{R}} (1 + |t|)^{\gamma'} \sup_{|s| \leq \delta_n} |Q_n'(t + s)|^2 \right\}
\]
\[
\leq C \delta_n^2 \mathbb{E} \left\{ \sup_{x \in \mathbb{R}} [(1 + |x|)^{\gamma'} |Q_n'(x)|^2} \right\} = O(\delta_n^2)
\]
and completes the proof. ♦

Remark 7. It is worthwhile to note that the modulus of continuity of \( G_n \) has the order \( \delta_n^{1-2/q} \), while that of \( Q_n \) has a higher order \( \delta_n \). ♦

9 Proof of Proposition 1

We shall adopt the truncation technique to deal with \( M_n \). For \( r > 0 \) define the function \( g1_{|>r} \) by \((g1_{|>r})(x) = g(x)1_{|x|>r}\) and

\[
M_n(g1_{|>r}) = n^{-1/2} \sum_{k=1}^{n} \{ g(X_k)1_{|X_k|>r} - \mathbb{E}[g(X_k)1_{|X_k|>r} | F_{k-1}] \}.
\] (80)

The function \( g1_{|\leq r} \) and the process \( M_n(g1_{|\leq r}) \) are similarly defined. Since \( M_n(g) = M_n(g1_{|>r}) + M_n(g1_{|\leq r}) \), the tightness of \( \{ M_n(g) : g \in \mathcal{G}_{\gamma, \mu} \} \) follows from Lemmas 11 and 12. To see this, for any \( \delta, \eta > 0 \), by Lemma 11, there exists \( r > 0 \) such that

\[
\limsup_{n \to \infty} \mathbb{P}^x \left\{ \sup_{g \in \mathcal{G}_{\gamma, \mu}} |M_n(g1_{|>r})| \geq \frac{\delta}{4} \right\} \leq \frac{\eta}{4}.
\] (81)

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By Lemma 12, there exists $U_1, \ldots, U_I$ with $I < \infty$ such that $1_{|x| \leq r}G_{\gamma, \mu} \subset \bigcup_{i=1}^I U_i$ and

$$\limsup_{n \to \infty} \mathbb{P}^* \left\{ \max_{1 \leq i \leq I} \sup_{g \in U_i} |M_n(g - h)| \geq \frac{\delta}{4} \right\} \leq \eta \frac{4}{4}.$$

(82)

Let $T_i = \{h + p1_{|x| > r} : h \in U_i, \ p \in G_{\gamma, \mu}, 1 \leq i \leq I\}$. Then $G_{\gamma, \mu} \subset \bigcup_{i=1}^I T_i$ and we have

$$\limsup_{n \to \infty} \mathbb{P}^* \left\{ \max_{1 \leq i \leq I} \sup_{g \in T_i} |M_n(g - h)| \geq \frac{\delta}{4} \right\} \leq \limsup_{n \to \infty} \mathbb{P}^* \left\{ \sup_{g \in G_{\gamma, \mu}} |M_n(g - h)| \geq \frac{\delta}{2} \right\} \leq \eta \frac{\eta}{2}$$

in view of (81) and (82). Thus by definition $\{M_n(g) : g \in G_{\gamma, \mu}\}$ is tight since $I < \infty$ and $\delta$ and $\eta$ are arbitrarily chosen.

The finite-dimensional convergence is a direct consequence of the martingale central limit theorem. The case when $G = H_{\eta, \delta}$ can be similarly proved.

Lemma 11. (i) Assume $\mathbb{E}(|X_1|^\gamma) < \infty, \ \gamma \geq 0$. Then

$$\lim_{t \to \infty} \limsup_{n \to \infty} \mathbb{E}^* \left\{ \sup_{g \in G_{\gamma, \mu}} |M_n(g1_{|x| > t})|^2 \right\} = 0.$$

(83)

(ii) Under conditions of (ii) of Proposition 1,

$$\lim_{t \to \infty} \limsup_{n \to \infty} \mathbb{E}^* \left\{ \sup_{g \in H_{\eta, \delta}} |M_n(g1_{|x| > t})| \right\} = 0.$$

(84)

Proof. (i) First assume $\mu < 1$. We shall generalize the argument in Giné and Zinn (1986).

Let $\alpha = (1 - \mu)^{-1}, \ g_r(x) = g(x)1_{x > r^\alpha}, \ r \in \mathbb{N}$, and the interval $I_j = (j^\alpha, (j + 1)^\alpha]$. Write $M_n(g_r) = A_n(g; r) + B_n(g; r)$, where

$$A_n(g; r) = n^{-1/2} \sum_{j=r}^{\infty} \sum_{k=1}^{n} \left\{ [g(X_k) - g(j^\alpha)]1_{X_k \in I_j} - \mathbb{E}[(g(X_k) - g(j^\alpha))1_{X_k \in I_j}|\mathcal{F}_{k-1}] \right\}$$
and

\[ B_n(g; r) = n^{-1/2} \sum_{j=r}^{\infty} \sum_{k=1}^{n} \{g(j^\alpha)1_{X_k \in I_j} - \mathbb{E}[g(j^\alpha)1_{X_k \in I_j} | \mathcal{F}_{k-1}]\}. \]  

(85)

Let \( Z_{j,n} = n^{-1/2} \sum_{k=1}^{n} [1_{X_k \in I_j} - \mathbb{E}(1_{X_k \in I_j} | \mathcal{F}_{k-1})] \). Then \( \|Z_{j,n}\|^2 \leq \mathbb{P}(X_k \in I_j) =: p_j \). By the Cauchy-Schwarz inequality and (59) of Lemma 3,

\[
\mathbb{E} \left\{ \sup_{g \in \mathcal{G}_{\gamma, \mu}} \sum_{j=r}^{\infty} |g(j^\alpha)Z_{j,n}| \right\}^2 \leq \mathbb{E} \left\{ \sup_{g \in \mathcal{G}_{\gamma, \mu}} \sum_{j=r}^{\infty} g^2(j^\alpha)(1 + j^\alpha)^{\gamma} \sum_{j=r}^{\infty} (1 + j^\alpha)^{\gamma} Z_{j,n}^2 \right\}
\]

\[ \leq C_{\gamma, \mu} \sum_{j=r}^{\infty} (1 + j^\alpha)^{\gamma} \mathbb{E}(Z_{j,n}^2) \leq C_{\gamma, \mu} \sum_{j=r}^{\infty} (1 + j^\alpha)^{\gamma} p_j. \]

Since \( B_n(g; r) = \sum_{j=r}^{\infty} g(j^\alpha)Z_{j,n} \) and \( \mathbb{E}[|X_1|^{1/\alpha} \gamma] = \mathbb{E}(|X_1|) < \infty \),

\[ \lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{E}\left[ \sup_{g \in \mathcal{G}_{\gamma, \mu}} |B_n(g; r)|^2 \right] \leq C_{\gamma, \mu} \limsup_{r \to \infty} \sum_{j=r}^{\infty} (1 + j^\alpha)^{\gamma} p_j = 0. \]

We now deal with \( A_n(g; r) \). Clearly (83) follows if the preceding inequality also holds for \( A_n(g; r) \). To this end, let \( S_n(u) = n^{-1/2} \sum_{k=1}^{n} [1_{X_k \in J(u)} - \mathbb{E}(1_{X_k \in J(u)} | \mathcal{F}_{k-1})] \), where \( J(u) = (u, \lfloor u^{1/\alpha} \rfloor + 1)^\alpha \). Since \( [g(X_k) - g(j^\alpha)]1_{X_k \in J(u)} = \int_{I_j} g'(u)1_{X_k \in J(u)} du \), we have

\[ |A_n(g; r)| \leq n^{-1/2} \sum_{j=r}^{\infty} \left| \sum_{k=1}^{n} \int_{I_j} g'(u)1_{X_k \in J(u)} - \mathbb{E}(1_{X_k \in J(u)} | \mathcal{F}_{k-1}) \right| du \]

\[ \leq \int_{\rho^\alpha}^{\infty} |g'(u)||S_n(u)| du. \]  

(86)

By the Cauchy-Schwarz inequality, \( |A_n(g; r)|^2 \leq \int_{\rho^\alpha}^{\infty} |S_n(u)|^2 w_{\gamma - \mu}(du) \) since \( g \in \mathcal{G}_{\gamma, \mu} \). So

\[ \limsup_{r \to \infty} \limsup_{n \to \infty} \mathbb{E}\left[ \sup_{g \in \mathcal{G}_{\gamma, \mu}} |A_n(g; r)|^2 \right] \leq \limsup_{r \to \infty} \int_{\rho^\alpha}^{\infty} \|S_n(u)\|^2 w_{\gamma - \mu}(du) \]

\[ \leq \limsup_{r \to \infty} \alpha \int_{\rho^\alpha}^{\infty} \mathbb{P}[t^\alpha < X_1 \leq (t + 1)^\alpha] (1 + t^\alpha)^{\gamma - \mu} t^{\alpha - 1} dt = 0 \]

in view of \( \mathbb{E}[(|X_k|^{1/\alpha} \alpha^{(\gamma - \mu) + \alpha - 1}] = \mathbb{E}(|X_k|) < \infty \). In the case that \( \mu = 1 \), let \( I_j = (2^j, 2^{j+1}] \) and \( J(u) = (u, 2u] \). It is easily seen that the above argument still works.

(ii) Let \( \mathcal{G} = \mathcal{H}_{\eta, \delta} \). By (85) and (86), if \( 0 \leq \eta - \delta < 1 \), then we have

\[ \mathbb{E}\left[ \sup_{g \in \mathcal{H}_{\eta, \delta}} |B_n(g; r)| \right] \leq \sum_{j=r}^{\infty} (1 + j^\alpha)^{\eta} \sqrt{p_j} \]
\[
\mathbb{E} \left[ \sup_{g \in \mathcal{H}_{\eta, \delta}} |A_n(g; r)| \right] \leq \int_{r^\alpha}^{\infty} (1 + u)^\delta \sqrt{\mathbb{P}[|X_1| \in J(u) \, \big| \, J]} du \\
\leq \int_{r^\alpha}^{\infty} (1 + v^\alpha)^\delta \sqrt{\mathbb{P}[v^\alpha \leq |X_1| \leq (v + 1)^\alpha]} (\alpha v^{\alpha - 1}) dv \\
\leq C_{\alpha, \delta} \int_{r^\alpha}^{\infty} v^{\alpha \eta} \sqrt{\mathbb{P}[v^\alpha \leq |X_1| \leq (v + 1)^\alpha]} dv
\]

which in view of (17) approaches zero if \( r \to \infty \). The case that \( \eta - \delta = 1 \) can be similarly dealt with.

\[\diamondsuit\]

**Lemma 12.** Let \( \mathcal{G} = \mathcal{G}_{\gamma, \mu} \) or \( \mathcal{H}_{\eta, \delta} \). Then for any \( r > 0 \), the process \( \{M_n(g1_{|g| \leq r}) : g \in \mathcal{G}\} \) is tight.

**Proof.** Consider first \( \mathcal{G} = \mathcal{G}_{\gamma, \mu} \). Recall (23) for the definition of the essential supremum norm \( d_2 \). Then \( d_2(g) \leq \|g\|_\infty := \sup_{x \in \mathbb{R}} |g(x)| \) and

\[
N(u, 1_{|\cdot| \leq r}\mathcal{G}_{\gamma, \mu}, d_2) \leq N(u, 1_{|\cdot| \leq r}\mathcal{G}_{\gamma, \mu}, \| \cdot \|_\infty).
\]

Let \( C_{\gamma, \mu} \) be the constant in (53) and define the Sobolev class

\[
\mathcal{S} = \left\{ h : [-r, r] \to \mathbb{R} : \sup_{x \in [-r, r]} |h(x)|^2 \leq C_{\gamma, \mu}(1 + r)^\gamma \text{and} \int_{-r}^{r} |h'(x)|^2 dx \leq (1 + r)^{\gamma - \mu} \right\}.
\]

Then there exists a constant \( C = C(r, \gamma, \mu) \) such that for every \( \epsilon > 0 \),

\[
\log N(\epsilon, \mathcal{S}, \| \cdot \|_\infty) \leq \frac{C}{\epsilon}.
\]

[cf. Birman and Solomjak (1967) or Theorem 2.7.1 in van der Vaart and Wellner (1996)].

For every \( g \in 1_{|\cdot| \leq r}\mathcal{G}_{\gamma, \mu} \), it is easily seen by (53) that \( \sup_{x \in [-r, r]} |g(x)|^2 \leq C_{\gamma, \mu}(1 + r)^\gamma \) and \( \int_{-r}^{r} |g'(x)|^2 dx \leq (1 + r)^{\gamma - \mu} \). Hence

\[
N(u, 1_{|\cdot| \leq r}\mathcal{G}_{\gamma, \mu}, \| \cdot \|_\infty) \leq N(u, \mathcal{S}, \| \cdot \|_\infty)
\]

and consequently

\[
\int_0^1 \sqrt{\log N(u, 1_{|\cdot| \leq r}\mathcal{G}_{\gamma, \mu}, d_2)} du \leq \int_0^1 \sqrt{\log N(u, \mathcal{S}, \| \cdot \|_\infty)} du < \infty.
\]
Therefore the lemma follows from Theorem 3.3 in Dedecker and Louhichi (2002) [see also Section 4.2 therein]. The case that $G = \mathcal{H}_{\eta,\delta}$ can be similarly proved. ♦

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