STATIONARY AND NON-GAUSSIAN DISTRIBUTIONS

\[ X_{t_i} = \begin{cases} 
\text{return } (\log S_{t_{i+1}} - \log S_{t_i}), \text{ or} \\
\text{volatility } (\sigma^2_{t_i}), \text{ or} \\
\text{interest rate } (r_{t_i}), \text{ etc}
\end{cases} \]

Stationary distribution of \( X_{t_i} \):

\[ F(x) = P(X_{t_i} \leq x) \quad \text{if same for all } i \]

or \( = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I(X_{t_i} \leq x) \quad \text{if meaningful} \)

or \( = \lim_{i \to \infty} P(X_{t_i} \leq x) \quad \text{if meaningful} \)

Returns of Geometric Brownian motion:

\[ X_{t_i} = \mu \Delta t_i + \sigma \Delta W_{t_i} \sim N(\mu \Delta t, \sigma^2 \Delta t) \]

Hence

\[ F = N(\mu \Delta t, \sigma^2 \Delta t) \]
If
\[ d \log S_t = \mu dt + \sigma dW_t \]
then (law of large numbers)
\[ \frac{\log S_t}{t} \Rightarrow \mu \]

\( \log S_t, S_t \) has no stationary distribution.
The Vasicek (or Ornstein-Uhlenbeck) Model

\[ dr_t = K(\alpha - r_t)dt + \sigma dW_t \]

Set \( u_t = r_t - \alpha \), get

\[ du_t = - Ku_t dt + \sigma dW_t \]

Itô:

\[ de^{Kt} u_t = e^{Kt} du_t + Ke^{Kt} u_t dt \]

\[ = e^{Kt} \sigma dW_t \]

\[ e^{Kt} u_t - e^{Ks} u_s = \sigma \int_s^t e^{Ku} dW_u \]

\[ = N \left( 0, \sigma^2 \int_s^t e^{2Ku} du \right) \]

\[ = N \left( 0, \frac{\sigma^2}{2K} (e^{2Kt} - e^{2Ks}) \right) \]

or

\[ u_t = e^{-K(t-s)} u_s + N \left( 0, \frac{\sigma^2}{2K} (1 - e^{-2K(t-s)}) \right) \]

Stationary distribution: \( t \to \infty \)

\[ u_t \to N \left( 0, \frac{\sigma^2}{2K} \right) \quad \text{and so} \quad r_t \to N \left( \alpha, \frac{\sigma^2}{2K} \right) \]

Also,

\[ \frac{1}{n} \sum_{i=1}^n I(u_{t_i} \leq x) \sim N \left( 0, \frac{\sigma^2}{2K} \right) (x) \]
In practice, to get stationary density $f = F'$:

- Collect $X_{t_i}$ over many periods $t_i \to t_{i+1}$
- Make histogram of result
APPROACHES TO NON GAUSSIAN STATIONARY DISTRIBUTIONS

- Jumps
- State dependent drift, volatility — for example:

\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \]

Stationary density

\[ f(x) = C \frac{1}{\sigma(x)^2} \exp \left( 2 \int_{x_0}^{x} \frac{\mu(y)}{\sigma^2(y)} dy \right) \]

- Stochastic volatility — for example

\[ dX_t = \mu_t dt + \sigma_t dW_t \]
\[ d\sigma_t^2 = K(\alpha - \sigma_t^2)dt + \gamma dB_t \]

perhaps correlation between \(dW, dB\): \(d[W, B]_t = \rho dt\)

Or Heston model
Jump Processes

*The simplest case: the Poisson process.*

$N_t$ is a Poisson process with rate $\lambda$ if:

- $N_t$ only evolves by jumping (and stays still between jumps), and the jump size is always 1
- $N_t - N_s$ is independent of $\mathcal{F}_u$ for $t \geq s \geq u$
- $N_0 = 0$
- $E(N_t) = \lambda t$

Some main properties of a Poisson process:

- $N_t - \lambda t$ is a martingale
- $P(\text{jump in } (t, t + \delta) \mid \mathcal{F}_t) = \lambda \delta + o_p(\delta)$
- $P(\text{more than one jump in } (t, t + \delta) \mid \mathcal{F}_t) = o_p(\delta)$
- the distribution of $N_t - N_s$ given $\mathcal{F}_u$ is Poisson($\lambda(t - s)$):

$$P(N_t - N_s = k \mid \mathcal{F}_u) = \frac{((\lambda(t - s))^k}{k!} \exp\{-\lambda(t - s)\}$$

Time varying (possibly random) $\lambda_t$ ("counting process"): 

- $N_t$ only evolves by jumping (and stays still between jumps), and the jump size is always 1
- $N_t - \int_0^t \lambda_u du$ is a martingale
- $N_0 = 0$
A frequent stock price model: the compound Poisson process:

\[ \log S_t - \log S_0 = \sum_{i=1}^{N_t} Z_i \]

More generally: jump-diffusion

\[ X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i \]

The Z’s (which are iid), N, W are independent
Itô’s formula

\[ df(X_t, t) = f'_x(X_{t-}, t) dX_t + f'_t(X_{t-}, t) dt \]
\[ + \frac{1}{2} f''_{xx}(X_{t-}, t) d[X^c, X^c]_t \]
\[ + \Delta f(X_t) - f'_x(X_{t-}) \Delta X_t \]

\[ X^c_t = \text{continuous part of } X_t, \text{ say,} \]
\[ = \mu t + \sigma W_t \]
\[ X_{t-} = \lim_{s \uparrow t} X_s \]
\[ \Delta X_t = X_t - X_{t-} \]
\[ = \text{jump at time } t, \text{ say,} \]
\[ = Z_i \text{ if } N_{t-} = i - 1, N_t = i \]
\[ dX_t = dX^c_t + \Delta X_t \]

where \( N_t \) is a Poisson process, and the \( Z_i \) are i.i.d. and independent of \( N_t \).
Itô’s formula on $S_t$ scale: $S_t = \exp(X_t)$

$$
\Delta \log S_t = \begin{cases} 
Z_i & \text{if jump} \\
0 & \text{otherwise}
\end{cases}
$$

Hence, if jump:

$$
\Delta S_t = S_t - S_{t-} \\
= S_{t-} \exp(Z_i) - S_{t-} \\
= (\exp(Z_i) - 1)S_{t-}
$$

$$
d \exp(X_t) = \exp(X_{t-})dX_t + \frac{1}{2} \exp(X_{t-})d[X^c, X^c]_t \\
+ \Delta \exp(X_t) - \exp(X_{t-})\Delta X_t
$$

$$
dS_t = S_{t-}dX_t + \frac{1}{2} S_{t-}\sigma^2 dt \\
+ \Delta S_t - S_{t-}\Delta X_t \\
= S_{t-}dX^c_t + \frac{1}{2} S_{t-}\sigma^2 dt + \Delta S_t
$$

$$
\frac{dS_t}{S_{t-}} = dX_t + \frac{1}{2}\sigma^2 dt + \frac{\Delta S_t}{S_{t-}} - \Delta X_t \\
= dX^c_t + \frac{1}{2}\sigma^2 dt + \frac{\Delta S_t}{S_{t-}} \\
= (\mu + \frac{1}{2}\sigma^2)dt + \sigma dW_t + \frac{\Delta S_t}{S_{t-}}
$$
HEDGING A SECURITY $V_t = f(S_t, t)$

Suppose $r = 0$

$$df(S_t, t) = f'_x(S_{t-}, t)dS_t \quad \text{hedgable}$$
$$+ (f'_t(S_{t-}, t) + \frac{1}{2}f''_s(S_{t-}, t)\sigma^2)dt \quad \text{will eliminate later}$$
$$+ \Delta f(S_t, t) - f'_s(S_{t-}, t)\Delta S_t$$

last line

$$= f(S_t, t) - f(S_{t-}, t) - f'_s(S_{t-}, t)\Delta S_t$$
$$= \frac{1}{2}f''_s(S_{t-}, t)\Delta S_t^2 + \ldots$$

(1)

not hedgable unless $f''_s(s, t) = 0$ or unless

$$\Delta S_t = J = (e^Z - 1) = \text{fixed if jump}$$

in which case

$$(\Delta S_t)^2 = \begin{cases} J^2 \text{ if jump} \\ 0 \text{ otherwise} \end{cases} = J\Delta S_t$$

and so on, so that:

$$(1) = \frac{1}{2}f''_s(S_{t-}, t)J\Delta S_t + \frac{1}{3!}f'''_s(S_{t-}, t)J^2\Delta S_t + \ldots$$
$$= g(S_{t-}, t)\Delta S_t$$
In this case

\[ df(S_t, t) = dS_t + dt \text{ terms} \]
\[ + g(S_{t-}, t) \Delta S_t \]

If two options \( f_i(S_t, t), i = 1, 2 \)

\[ df_1(S_t, t) - \frac{g_1(S_{t-}, t)}{g_2(S_{t-}, t)} df_2(S_t, t) \]
\[ = dS_t + dt \text{ terms}. \]

\( dt \) terms must be zero under any risk neutral measure

One option completes market.

In general: \( p \) Brownian motions, \( q \) jump sizes:

need \( p + q - 1 \) options to complete market.
Variation processes for jump processes. For processes of this type, $\langle \cdot, \cdot \rangle$ is not the same as $[\cdot, \cdot]$. We use the following definitions: $[X, \cdots, X]$ is given by

$$[X, \cdots, X]_t = \lim_{p \text{ times}} \sum (X_{t_{i+1}} - X_{t_i})^p$$

$$= \begin{cases} 
X_t & \text{if } p = 1 \\
\lim \sum (X_{t_{i+1}} - X_{t_i})^2 & \text{if } p = 2 \\
\sum \Delta X_{i}^p & \text{if } p > 2 
\end{cases}$$

On the other hand, $\langle X, \cdots, X \rangle$ is the predictable component in the Doob-Meyer decomposition of $[X, \cdots, X]$:

$$[X, \cdots, X] = \langle X, \cdots, X \rangle + \text{martingale}.$$ 

Important examples of variation processes. For the Poisson process,

$$[N, \cdots, N]_t = N_t$$

and

$$\langle N, \cdots, N \rangle_t = \lambda t.$$ 

For the compound Poisson:

$$[\log S, \cdots, \log S]_t = \sum_{i=1}^{N_t} Z_i^p$$
and

\[ \langle \log S, \cdots, \log S \rangle_t = E(Z^p) \lambda t. \]

From Itô’s formula:

\[
df(X_t) = f'(X_{t-})dX_t + \frac{1}{2} f''(X_{t-})d[X^c, X^c]_t \\
+ \Delta f(X_t) - f_x'(X_{t-}) \Delta X_t \\
= f'(X_{t-})dX_t + \frac{1}{2} f'(X_{t-})d[X, X]_t \\
+ \frac{1}{3!} f'''(X_{t-}) d[X, X, X]_t + \ldots
\]

since

\[
\Delta f(X_t) = f(X_t) - f(X_{t-}) \\
= f'(X_{t-}) \Delta X_t + \frac{1}{2} f''(X_{t-})(\Delta X_t)^2 \\
+ \frac{1}{3!} f'''(X_{t-})(\Delta X_t)^3 + \ldots
\]

and

\[
d[X, X]_t = d[X^c, X^c]_t + (\Delta X_t)^2
\]

This is the Taylor expansion interpretation of Itô’s formula.