MARTINGALES AND LOCAL MARTINGALES

If $S_t$ is a (discounted) security, the discounted P/L

$$V_t = \int_0^t \theta_u dS_u$$

need not be a martingale.

Can $V_t$ be a valid P/L? When?
ARBITRAGE WITH
STOCHASTIC INTEGRALS: AN EXAMPLE

Stock price: \( dS_t = \sigma S_t dW_t \)

Trading strategy: \( \theta_t = \frac{1}{\sigma S_t \sqrt{T-t}} \)

P/L: \( V_t = \int_0^t \frac{1}{\sigma S_u \sqrt{T-u}} dS_u \)

\[ = \int_0^t \frac{1}{\sqrt{T-u}} dW_u \]

Time change: set \( V_t = B_{F(t)} \)

Determination of \( F \):

\[ [B, B]_{F(t)} = [V, V]_t = \int_0^t \frac{1}{T-u} du = \log \left( \frac{T}{T-t} \right) \]

Choose: \( F(t) = \log \left( \frac{T}{T-t} \right) \), so \( [B, B]_t = t \)

Since \( V_t \) martingale on \([0, T)\), and \( F(0) = 0 \), \( F(T) = \infty \):

\( (B_t)_{0 \leq t < \infty} \) is standard Brownian motion (Levy)
Investment scheme:

\[ P/L: V_t = B_{F(t)} \text{ with } F(t) = \log \left( \frac{T}{T-t} \right) \]

We seek arbitrage profit of \( \alpha > 0 \) dollars

Stopping rule: \( \tau = \min\{t \geq 0 : V_t \geq \alpha\} \)

\[ \tau = F^{-1}(\tau') \text{ where } \tau' = \min\{t \geq 0 : B_t \geq \alpha\} \]

Since \( B \) is Brownian motion: \( P(\tau' < \infty) = 1 \)

And so \( P(\tau < T) = 1 \)

Modified P/L:

\[ V_{t \wedge \tau} = \int_0^t \theta_u I_{\{u \leq \tau\}} dS_u \]

**THIS TRADING STRATEGY EARS PROFIT \( \alpha \) WITH PROBABILITY 1 BY TIME \( T \)**

**WHAT IS WRONG?**

Relates to absence of credit constraint, as in 390 Lecture 5, p. 15-16
STANDARD WAY OF COPING WITH THE PROBLEM

1. THE CONCEPT OF LOCAL MARTINGALE (LOC MG):

$M_t$ is a local martingale for $t \in [0, T]$ if

there is a sequence $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ of stopping times so that $P(\tau_n = T) \to 1$ as $n \to \infty$, and so that $M_{t \wedge \tau_n}$ is a martingale for each $n$.

2. INVARIANCE OF LOC MGness UNDER STOCHASTIC INTEGRATION:

If $M_t$ is a continuous loc MG, if $\theta_t$ ia adapted, if

$\int_0^T \theta_t^2 d[M,M]_t < \infty$ with probability 1, then $\int_0^t \theta_u dM_u$ is a continuous loc MG.

(This is the real theorem about stochastic integrals being martingales)

3. THE IMPACT OF A CREDIT CONSTRAINT:

If $M_t$ is a loc MG, and $M_t \geq -K$ for all $t$ with probability 1, then $M_t$ is a supermartingale.

TRANSLATION: with a credit constraint, you cannot earn arbitrage profit, but you can have arbitrage loss.
EXPLANATION OF TRANSLATION

If $M_t$ is a supermartingale, then by Doob-Meyer:

$$M_t = N_t - D_t = \text{martingale} - \text{dividend}$$

The dividend, if any, is the arbitrage loss

Arbitrage loss (failure of individual trader) is more palatable assumption than arbitrage profit (failure of market)

RELATIONSHIP TO OUR EXAMPLE

$V_{\tau \land t}$ cannot be bounded below. Strategy requires infinite credit

PROOF OF ITEM 3:

Use Fatou’s lemma: if $X_n \to X$ a.s., and $X_n \geq K$ a.s. for all $n$, then $\liminf_{n \to \infty} E(X_n|\mathcal{G}) \geq E(X|\mathcal{G})$

Application here: $s < t$, $X_n = M_{t \land \tau_n}$, $X = M_t$, $\mathcal{G} = \mathcal{F}_s$:

$$M_{s \land \tau_n} = E(M_{t \land \tau_n}|\mathcal{F}_s) \quad \text{(optional stopping)}$$

and so, as $n \to \infty$:

$$M_s = \lim M_{s \land \tau_n} = \lim E(M_{t \land \tau_n}|\mathcal{F}_s) \geq E(\lim M_{t \land \tau_n}|\mathcal{F}_s) = E(M_t|\mathcal{F}_s)$$

Q.E.D.
QUANTILE HEDGING

(based on paper by Föllmer and Leukert, Sect 1-3)

SETTING: STOCK PRICE PROCESS $X_t$, PAYOFF $H \geq 0$ AT TIME $T$, COMPLETE MARKET ($P^*$ exists and is unique), $r = 0$

Hedging strategy: $V_t = V_0 + \int_0^t \xi_s dX_s$; admissible: $V_t \geq 0$

Solvency set: $A = \{V_T \geq H\}$

THE PROBLEM: GAMBLING INTELLIGENTLY

Initial capital: $\tilde{V}_0 < E^*(H)$. PROBLEM: find $\xi$ to maximize $P(A)$ (= the probability of staying out of jail?) subject to starting value $V_0 \leq \tilde{V}_0$

ASPECT OF SOLUTION

Enough to hedge $HI_A$. If you spend money to hedge payoffs outside $A$, you are misallocating resources if the goal is $P(A) = \max$.

If instead constraint $V_t \geq -K$, then you wish to end with $V_T = -K$ outside $A$. You seek to hedge $HI_A - KI_{A^c}$ ($A^c$ is the complement of $A$)
Formalization as proposition (2.8): Suppose $\tilde{A}$ solves $P(A) = \max$ subject to $E^*[HI_A] \leq \tilde{V}_0$. Let $\tilde{\xi}$ be the perfect hedge for payoff $HI_A$:

$$HI_{\tilde{A}} = E^*[HI_A] + \int_0^T \tilde{\xi}_s dX_s$$  \hspace{1cm} (2.11)

Then $\tilde{V}_t = \tilde{V}_0 + \int_0^t \tilde{\xi}_s dX_s$ solves the original optimization problem, and \{\$\tilde{V}_T \geq H\} = \tilde{A}$ up to set of proba zero.

Proof: (1) Let $V_t$ be any admissible strategy, $V_0 \leq \tilde{V}_0$. $A$ is success set for this $V$. Then $P(A) \leq P(\tilde{A})$ (2.14)

Subproof: $V_T \geq HI_A$ since $V_T \geq 0$. Since $V_t$ is superMG:

$$\tilde{V}_0 \geq V_0 \geq E^*[V_T] \geq E^*[HI_A]$$

Hence $A$ satisfies the constraint in the theorem. Hence (2.14)

(2) Let $V_t = V_0 + \int_0^t \tilde{\xi}_s dX_s$ for $\tilde{V}_0 \geq V_0 \geq E^*[HI_{\tilde{A}}]$

$$V_t \geq E^*[HI_A] + \int_0^T \tilde{\xi}_s dX_s = E^*[HI_A | \mathcal{F}_t] \geq 0$$

so $V_t$ is admissible
$V_t$ is optimal

From (2.11), and since $V_0 \geq E^*[HI_{\tilde{A}}]$:

$$HI_{\tilde{A}} = E^*[HI_{\tilde{A}}] + \int_0^T \tilde{\xi}_s dX_s \leq V_0 + \int_0^T \tilde{\xi}_s dX_s = V_T$$

If we set $A = \{V_T \geq H\}$, then: $\tilde{A} \subseteq A$ a.s.

But from (2.14): $P(A) \leq P(\tilde{A})$, so $\tilde{A} = A$ a.s.

In particular: $\tilde{V}_t$ is optimal. QED.
NEXT PROBLEM: HOW TO SOLVE
THE PROBLEM FROM PROPOSITION (2.8)?

\[ P(A) = \max, \text{ subject to } E^*[H I_A] \leq \tilde{V}_0 \]

Define \( Q^* : \frac{dQ^*}{dP^*} = \frac{H}{E^*[H]} = \frac{H}{H_0} \)

Rewrite problem:

\[ P(A) = \max, \text{ subject to } Q^*(A) \leq \alpha \]

where \( \alpha = \frac{\tilde{V}_0}{H_0} \). Note: \( Q^* \ll P, P^* \), but maybe not reverse

SOLUTION TO THIS PROBLEM:
THE NEYMAN-PEARSON LEMMA

\[ \tilde{A} = \left\{ \frac{dP}{dP^*} > c \frac{dQ^*}{dP^*} \right\} \cup \left\{ \frac{dP}{dP^*} = c \frac{dQ^*}{dP^*} \right\} \cap B \]

where \( c \) is constant and \( B \) is outcome of coin toss independent of everything else, so that

\[ \alpha = Q^*(\tilde{A}) = Q^* \left\{ \frac{dP}{dP^*} > c \frac{dQ^*}{dP^*} \right\} + Q^* \left\{ \frac{dP}{dP^*} = c \frac{dQ^*}{dP^*} \right\} Q^*(B) \]

\( B \) is only relevant if \( Q^* \left\{ \frac{dP}{dP^*} = c \frac{dQ^*}{dP^*} \right\} \neq 0 \)
SPECIFIC DEFINITION

\[ \tilde{c} = \min \left\{ c : Q^* \left\{ \frac{dP}{dP^*} \geq c \frac{dQ^*}{dP^*} \right\} \leq \alpha \right\} \]

\( Q^*(B) \) is defined to reach equality \( \alpha = Q^*(\tilde{A}) \)

Further mathematical formulation yields

\( P(B) = P^*(B) = Q^*(B) \)

ALTERNATIVE DESCRIPTION

\[ \frac{dQ^*}{dP^*} = \frac{H}{E^*[H]} = \frac{H}{H_0} \]

and so

\[ \tilde{A} = \left\{ \frac{dP}{dP^*} > c \frac{H}{E^*[H]} \right\} \cup \left\{ \frac{dP}{dP^*} = c \frac{H}{E^*[H]} \right\} \cap B \]
THE “DUAL” PROBLEM

Find $V_0 = \min$, subject to $P(V_T \geq H) \geq 1 - \epsilon$

This is the same as:

Find $E^*[HI_A] = \min$, subject to $P(A) \geq 1 - \epsilon$

(We assume no liability if you actually use this...)

Solution set $\tilde{A}$ has the same Neyman-Pearson form as for the “primal” problem, but rewrite, with $b = 1/c$:

$$\tilde{A} = \left\{ \frac{dP}{dP^*} > c \frac{dQ^*}{dP^*} \right\} \cup \left\{ \frac{dP}{dP^*} = c \frac{dQ^*}{dP^*} \right\} \cap B$$

$$= \left\{ \frac{dQ^*}{dP} > b \right\} \cup \left\{ \frac{dQ^*}{dP} = b \right\} \cap B$$

(Recall: $Q^* \ll P$)

Determine $b$ and $P(B)$ from $P(\tilde{A}) = 1 - \epsilon$

When calculating $\tilde{V}_0 = E^*[HI_{\tilde{A}}]$, use $P^*(B) = P(B)$
THE BLACK SCHOLES MODEL

\[ dX_t = mX_t dt + \sigma X_t dW_t \]

and, as usual,

\[ \log(X_T) = \log(x_0) + \sigma W_T + (m - \frac{1}{2}\sigma)T \]

Risk neutral measure:

\[
\frac{dP^*}{dP} = \exp \left\{ -\frac{m}{\sigma} W_T - \frac{1}{2} \left( \frac{m}{\sigma} \right)^2 T \right\} = \text{const} \times X_T^{-m/\sigma^2} \\
\left( \text{const} = x_0^{m/\sigma^2} \exp \left\{ \frac{1}{2} \frac{m}{\sigma^2} (m - \sigma)T \right\} \right) \]
CALL PAYOFF: $H = (X_T - K)^+$

\[
\frac{dP^*}{dP} = \text{const} \times X_T^{-m/\sigma^2}
\]

\[
\tilde{A} = \left\{ \frac{dP}{dP^*} > c \frac{H}{E^*[H]} \right\} \cup \left\{ \frac{dP}{dP^*} = c \frac{H}{E^*[H]} \right\} \cap B
\]

Since

\[
P \left\{ \frac{dP}{dP^*} = c \frac{H}{E^*[H]} \right\} = 0
\]

can take (for some $\lambda$)

\[
\tilde{A} = \left\{ X_T^{-m/\sigma^2} > \lambda H \right\}
\]

PRIMAL PROBLEM:

$\tilde{A}$ maximizes success probability subject to initial capital $\tilde{V}_0$ if $\lambda$ is such that $E^*[HI_{\tilde{A}}] = \tilde{V}_0$
SOLUTION

\[ \tilde{A} = \left\{ X_T^{-m/\sigma^2} > \lambda(X_T - K)^+ \right\} \quad \text{and} \quad E^*[(X_T - K)^+I_{\tilde{A}}] = \tilde{V}_0 \]

CASE (i): \( m \leq \sigma^2 \)

(the other case is similar)

If \( X_T \leq K \), then \( X_T \in \tilde{A} \)

If \( X_T > K \), then \( x \to x^{-m/\sigma^2} - \lambda(x - K) \) is decreasing, so \( x^{-m/\sigma^2} > \lambda(x - K) \) is the same as \( x < c \) for some \( c \)

Obviously, \( c \geq K \), and so

\[ \tilde{A} = \{X_T < c\} = \{W_T^* < b\} \]

Since \( \log(X_T) = \log(x_0) + \sigma W_T^* - \frac{1}{2}\sigma^2 T \)

and by \( c = x_0 \exp\{\sigma b - \frac{1}{2}\sigma^2 T\} \)

Since \( W_T^* = W_T + \frac{m}{\sigma} T \):

Success probability: \( P(\tilde{A}) = \Phi\left( \frac{b - m}{\sigma \sqrt{T}} \right) \)
Initial capital

Modified option:

$$(X_T - K)^+ I_A = (X_T - K)^+ I_{\{X_T > c\}}$$

$$= (X_T - c)^+ + (c - K) I_{\{X_T > c\}}$$

$$= (X_T - c)^+ + (c - K) I_{\{W_T^* > b\}}$$

And so

$$\tilde{V}_0 = \text{BS-price of call with strike } c$$

$$+ (c - K) \times \text{BS-price of binary option}$$

$$= \text{BS-price of call with strike } c$$

$$+ (c - K) \Phi \left( -\frac{b}{\sqrt{T}} \right)$$
PRIMAL AND DUAL PROBLEM

PRIMAL PROBLEM:

Start with $\tilde{V}_0$, find $c$ or $b$, then compute success probability $P(\tilde{A})$

DUAL PROBLEM:

Start with $P(\tilde{A})$

Find $b$ from $P(\tilde{A}) = \Phi\left(\frac{b-m}{\sigma T}\right)$

Find $c = x_0 \exp\{\sigma b - \frac{1}{2} \sigma^2 T\}$

Compute $\tilde{V}_0 = \text{BS-price of call with strike } c+(c-K)\Phi(-\frac{b}{\sqrt{T}})$

Example in paper (bottom of p. 261): cost reduction of 41% for failure probability of 5% 

Quite possibly illegal

Risk management by interval constraints can be subverted:

For any constraint on $P(\tilde{A})$ (=, say, 95%), $c$ is given, and

$\tilde{V}_0 \to 0$ as $K \to c$