AMERICAN OPTIONS

REVIEW OF STOPPING TIMES

\( \tau \) is stopping time if \( \{ \tau \leq t \} \in \mathcal{F}_t \) for all \( t \)

Important example: the first passage time for continuous process \( X \):

\[
\tau_m = \min \{ t \leq 0 : X(t) = m \}
\]

\( (\tau_m = \infty \) if \( X(t) \) never takes the value \( m \) This is a stopping time (see Sect 8.2 of Shreve)

Optional sampling theorem:

Let \( \tau \) be a stopping time. If \( (M_t) \) is a martingale (submartingale, supermartingale), then \( (M_{\tau \wedge t}) \) is a martingale (submartingale, supermartingale)
THE PERPETUAL AMERICAN PUT

Process: \( dS_t = rS_t dt + \sigma S_t dW_t \)

\( T \) = all stopping times that take value in \([0, \infty]\)

Value of perpetual put:

\[
v_*(x) = \max_{\tau \in T} E^*[e^{-r\tau} (K - S_\tau)^+ I_{\{\tau < \infty\}} \mid S_0 = x] \\
= \max_{\tau \in T} E^*[e^{-r\tau} (K - S_\tau) I_{\{\tau < \infty\}} \mid S_0 = x]
\]

The last equality: no point in exercising if \( S_t > K \)

Reasonable to exercise when \( S_t \) hits some level \( L_* \)

What is \( L_* \)?

What is the price \( v_*(x) \)?
INTERMEDIATE PROBLEM

Hitting time: \( \tau_L = \min\{t \leq 0 : S(t) = L\} \)

\[
v_L(x) = E^*[e^{-r\tau_L}(K - S_{\tau_L})I_{\{\tau_L < \infty\}} \mid S_0 = x]
\]

Program: find \( v_L(x) \), then find value \( L_* \) which maximizes \( v_L(x) \)

This is simpler, since \( v_L(x) \) does not involve any maximization over stopping times

Since \( S_{\tau_L} = L \) on \( \{\tau_L < \infty\} \):

\[
v_L(x) = E^*[e^{-r\tau_L}(K - L)I_{\{\tau_L < \infty\}} \mid S_0 = x]
\]

which depends only on distribution of \( \tau_L \)

One derivation for this price: Shreve Section 8.3.1 (more elegant)

We shall use: Winter Lecture 1 (brute force)
Set

\[ X_t = \log S_t - \log S_0 \text{ and } \nu = r - \frac{1}{2} \sigma^2 \]

Then

\[ X_t = \nu t + \sigma W_t \]

Hitting time

\[ \tau = \min \{ t : X_t = b \} \]

From Lecture 1 (p. 16): density of \( \tau \):

\[ f_\tau(t) = \frac{|b|}{\sqrt{2\pi} \sigma^2 t^3} \exp \left( -\frac{b^2}{2\sigma^2 t} + \frac{\nu}{\sigma^2} b - \frac{1}{2} \frac{\nu^2}{\sigma^2} t \right) \]

Since \( S_0 = x \):

\[ S_{\tau_L} = L \leftrightarrow X_{\tau_L} = \log(L) - \log(x) \]

And so \( \tau = \tau_L \) if

\[ b = \log(L) - \log(x) = -\log(x/L) \]
BACK TO ORIGINAL PROBLEM

For $x \geq L$:

$$v_L(x) = E^*[e^{-r\tau_L}(K - L)I_{\{\tau_L<\infty\}} \mid S_0 = x]$$

$$= \int_0^\infty e^{-rt}(K - L)f(t)dt$$

$$= (K - L) \exp \left( \frac{2rb}{\sigma^2} \right)$$

$$= (K - L) \left( \frac{x}{L} \right)^{-\frac{2r}{\sigma^2}}$$

Obviously, for $x \leq L$: $v_L(x) = (K - x)$ (exercise immediately)
OPTIMAL VALUE OF $L$

$$v_L(x) = \begin{cases} (K - L)L^{2r} \times x^{-\frac{2r}{\sigma^2}} & \text{for } x \geq L \\ (K - x) & \text{otherwise} \end{cases}$$

For fixed $x$: maximize with respect to $L$:

$$\frac{\partial}{\partial L} (K - L)L^{2r} \sigma^{-2} = -L^{2r} \sigma^{-2} + \frac{2r}{\sigma^2} (K - L)L^{2r} \sigma^{-2} - \frac{1}{2}$$

$$= -\frac{2r + \sigma^2}{\sigma^2} L^{2r} \sigma^{-2} + \frac{2r}{\sigma^2} KL^{2r} \sigma^{-2} - 1$$

$$\frac{\partial}{\partial L} (K - L)L^{2r} \sigma^{-2} = 0 \iff L = L_*$$

where

$$L_* = \frac{2r}{2r + \sigma^2} K$$

This corresponds to a maximum of $(K - L)L^{2r} \sigma^{-2}$ since only stationary point and since $(K - L)L^{2r} \sigma^{-2} = 0$ for $L = 0$ and $(K - L)L^{2r} \sigma^{-2} \to -\infty$ as $L \to \infty$. 
ANALYTIC CHARACTERIZATION OF PUT PRICE

\[ v_{L*}(x) = \begin{cases} (K - L_*)(x/L_*)^{-\frac{2r}{\sigma^2}} & \text{for } x \geq L_* \\ (K - x) & \text{otherwise} \end{cases} \]

Hence

\[ v'_{L*}(x) = \begin{cases} -(K - L_*) \frac{2r}{\sigma^2 x} (x/L_*)^{-\frac{2r}{\sigma^2}} & \text{for } x \geq L_* \\ -1 & \text{otherwise} \end{cases} \]

Right derivative at \( L_* \):

\[ v'_{L*}(L_*+) = -(K - L_*) \frac{2r}{\sigma^2 L_*} = -1 \]

\( v'_{L*}(x) \) is continuous at \( L_* \): “smooth pasting”
CAN VERIFY DIRECTLY THAT

(i) $v_{L*}(x) \geq (K - x)^+$ for all $x \geq 0$

(ii) $rv_{L*}(x) - r x v'_{L*}(x) - \frac{1}{2} \sigma^2 x^2 v''_{L*}(x) \geq 0$ for all $x \geq 0$

(iii) for each $x \geq 0$, one of (i) or (ii) is an equality

(i)-(iii) (complementarity conditions) determine $v_{L*}(x)$

TRADING INTERPRETATION

$$d[e^{-rt}v_{L*}(S_t)]$$

$$= e^{-rt} \left[ -rv_{L*}(S_t)dt + v'_{L*}(S_t)dS_t + \frac{1}{2}v''_{L*}(S_t)d[S, S]_t \right]$$

$$= v'_{L*}(S_t)d\tilde{S}_t - d\tilde{D}_t$$

where, since $d\tilde{S}_t = d[e^{-rt}S_t] = -re^{-rt}S_tdt + e^{-rt}dS_t$:

$$d\tilde{D}_t = e^{-rt} \left[ rv_{L*}(S_t)dt - rv'_{L*}(S_t)S_t - \frac{1}{2}v''_{L*}(S_t)d[S, S]_t \right]$$

$$= e^{-rt} \left[ rv_{L*}(S_t)dt - rv'_{L*}(S_t)S_tdt - \frac{1}{2}v''_{L*}(S_t)\sigma^2 S^2_t dt \right]$$

$$\geq 0 \text{ by (ii)} \quad (ii)'$$

(i) + (ii)' means that $v'_{L*}(S_t)$ is a superreplication of the American option
STRUCTURE OF THE DIVIDEND

Precise form of (ii) (for all \( x \geq 0 \))

\[
r v_{L^*}(x) - r x v'_{L^*}(x) - \frac{1}{2} \sigma^2 x^2 v''_{L^*}(x) = \begin{cases} 
0 & \text{if } x > L_* \\
rK & \text{if } x < L_* 
\end{cases}
\]

Hence

\[
d\tilde{D}_t = e^{-rt} \left[ r v_{L^*}(S_t) - r v'_{L^*}(S_t)S_t - \frac{1}{2} v''_{L^*}(S_t)\sigma^2 S_t^2 \right] dt \\
= -e^{-rt} rK I\{S_t < L_*\} dt
\]

FINANCIAL INTERPRETATION

The hedging strategy pays a dividend of \( rK \) \$ for when \( S_t < L_* \). This is arbitrage profit if the owner of the option does not exercise at time \( \tau_{L_*} \).

PROBABILISTIC INTERPRETATION

\( M_t = e^{-rt} v_{L^*}(S_t) \) is a supermartingale with Doob-Meyer decomposition \( v'_{L^*}(S_t)d\tilde{S}_t - d\tilde{D}_t \)

\( M_{\tau_{L^*} \wedge t} \) is a martingale
FINALLY: \( v_{L*}(x) = v_*(x) \)

Proof: Since \( e^{-rt}v_{L*}(S_t) \) is a supermartingale:

For any stopping time \( \tau \in \mathcal{T} \)

\[
\begin{align*}
v_{L*}(x) &= v_{L*}(S_0) \\
&\geq E^*[e^{-r(\tau\land t)}v_{L*}(S_{\tau\land t})] \\
&\rightarrow E^*[e^{-r\tau}v_{L*}(S_\tau)] \quad (\text{as } t \rightarrow \infty)
\end{align*}
\]

and so

\[
v_{L*}(x) \geq \max_{\tau \in \mathcal{T}} E^*[e^{-r\tau}v_{L*}(S_\tau)] = v_*(x)
\]

On the other hand, since \( \tau_{L*} \in \mathcal{T} \)

\[
\begin{align*}
v_*(x) &= \max_{\tau \in \mathcal{T}} E^*[e^{-r\tau}v_{L*}(S_\tau)] \\
&\geq E^*[e^{-r\tau_{L*}}v_{L*}(S_{\tau_{L*}})] \\
&= v_{L*}(x)
\end{align*}
\]

The equality follows
THE REGULAR AMERICAN PUT

Exercise time $\tau$ must be $\leq T$

$$v(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} E^* [e^{-r(\tau-t)}(K - S_{\tau})^+ | S_t = x]$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times taking values in $[t, T]$
ANALYTIC CRITERIA
FROM TRADING INTERPRETATION

Solvency requires: \( v(t, S_t) \geq (K - S_t)^+ \), or

(i) \( v(x, t) \geq (K - x)^+ \)

Replication considerations. Ito’s formula:

\[
d[e^{-rt}v(t, S_t)] = v_x(t, S_t)dS_t - d\tilde{D}_t
\]

Where \( d\tilde{D}_t =
\[
e^{-rt} \left[ rv(t, S_t) - v_t(t, S_t) - rS_t v_x(t, S_t) - \frac{1}{2} \sigma^2 S_t^2 v_{xx}(t, S_t) \right] dt
\]

Superreplication requires: \( d\tilde{D}_t \geq 0 \), or

(ii) \( rv(t, x) - v_t(t, x) - rxv_x(t, x) - \frac{1}{2} \sigma^2 x^2 v_{xx}(t, x) \geq 0 \)

Getting the lowest price:

(iii) for each \( x \geq 0 \), one of (i) or (ii) is an equality

(otherwise one could lower \( v(x,t) \) and still have a solvent superreplication)

(i)-(iii): the complementarity conditions again
STOPPING AND CONTINUATION REGIONS

Stopping region: \( S = \{(t, x) : v(t, x) = (K - x)^+\} \)

Continuation region: \( C = \{(t, x) : v(t, x) > (K - x)^+\} \)

Rationale: If \((t, S_t) \in C\): option is worth more than exercise value. Keep it.

On the other hand, if \(v(t, x) = (K - x)^+\):

\[
rv(t, x) - v_t(t, x) - rxv_x(t, x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rK
\]

Hence if \((t, S_t) \in S\):

\[
d\tilde{D}_t = e^{-rt} \left[ rv(t, S_t) - v_t(t, S_t) - rS_t v_x(t, S_t) - \frac{1}{2}\sigma^2 S_t^2 v_{xx}(t, S_t) \right] dt = rK dt
\]

You’re being arbitraged. Get rid of it.
STRUCTURE OF THE DIVIDEND

\[ rv(t, x) - v_t(t, x) - rv_x(t, x) - \frac{1}{2} \sigma^2 x^2 v_{xx}(t, x) = \begin{cases} 0 & \text{for } (t, x) \in \mathcal{C} \\ rK & \text{for } (t, x) \in \mathcal{S} \end{cases} \]

Hence, as before:

\[ d\tilde{D}_t = rKI_{\{(t, S_t) \in \mathcal{S}\}} dt \]

STOPPING RULE

\[ \tau_* = \min\{t \in [0, T] : (t, S_t) \in \mathcal{S}\} \]

\[ M_t = e^{-rt}v(S_t, t) \text{ is a supermartingale with Doob-Meyer decomposition } v_x(S_t, t)d\tilde{S}_t - d\tilde{D}_t \]

\[ M_{\tau_* \wedge t} \text{ is a martingale} \]
STOPPING BOUNDARY

Stopping region: \( \mathcal{S} = \{(t, x) : v(t, x) = (K - x)^+\} \)

Continuation region: \( \mathcal{C} = \{(t, x) : v(t, x) > (K - x)^+\} \)

Boundary: \( x = L(T - t) \):

\( (t, x) \in \mathcal{S} \) iff \( x \leq L(T - t) \)

\( (t, x) \in \mathcal{C} \) iff \( x > L(T - t) \)

“smooth pasting” continues to hold:

\( v_x(x+, t) = v_x(x-, t) = -1 \) for \( x = L(T - t) \), for \( t < T \)
AMERICAN CALL OPTIONS
CASE OF NO DIVIDEND

The calculations from the discrete case (Autumn Lecture 5) carry over.

The value is the same as for European options

WITH DIVIDEND AT DISCRETE TIMES
Reduces to a discrete time problem. Between dividend times, reduces to a European options problem