1. **Implied volatilities.**

We are concerned with a discounted stock \( \tilde{S} = S / \Lambda \). As in class, we also use a time changed instrument of the \( P^* \) form \( dS_t = \tilde{S}_t dW_t \), and a stopping time \( \tau \) is of the form

\[
\tau = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_p \wedge \Xi^+, \]

where

\[
\tau_i = \inf \{ t \geq t_i : \tilde{S}_t = K_i \} \]

and \( t_1, ..., t_p \leq \Xi^+ \). Suppose that \( E^*(\tilde{S}_\tau - K_i)^+ \) equals the discounted market price at time zero for a call option with strike price \( K_i \). Also suppose that there are indices \( i_1 \) and \( i_2 \) so that

\[
t_{i_1} \leq t_{i_2} \leq t_i, \ i \neq i_1, i_2 \]

(a) Show that \( t_{i_1} \) equals the implied (cumulative) volatility \( \Xi_i \), of the traded option with strike price \( K_{i_1} \).

(b) Determine if or when \( t_{i_2} = \Xi_{i_2} \). If or when the equality is not satisfied, determine when \( t_{i_2} < \Xi_{i_2} \), and when \( t_{i_2} > \Xi_{i_2} \).

(c) If or when \( t_{i_2} \neq \Xi_{i_2} \), discuss whether there is a rationale for thinking of \( t_{i_2} \) as an implied volatility.

2. **Hedging strategies.**

We let \( \sigma_t \) be the instantaneous volatility of \( \tilde{S} = S / \Lambda \). Consider the bound

\[
\Xi^- \leq \int_0^T \sigma_u^2 du \leq \Sigma^+. \tag{1} \]

We are dealing with a European payoff \( f(S_T) \) at time \( T \).

(a) Suppose \( f \) is concave. State the discounted ask price \( \tilde{A} \) at time \( t = 0 \). Then derive the hedging strategy that covers the payoff under starting value \( \tilde{A} \) and bound (1), and give the portfolio value process \( V_t \) and dividend process \( D_t \), if any.

(b) Answer the same question as in (a) for when \( \Xi^- = 0 \) and \( s \rightarrow f(s) \) is convex for \( 0 \leq s < s_0 \), and concave for \( s_0 < s \). \( f \) is assumed to be continuous. Also explain what the optimal stopping time \( \tau \) is in the price \( \sup_{0 \leq \tau \leq \Xi^+} E^* f(\tilde{S}_\tau) \).