
Homework 6. Due Wed November 17.

1. **The Black-Scholes-Merton formula.** Derive the B-S-M formula for the price of a call option according to p. 19 in the lecture notes. (cf. Exercise 3.5 (p. 118) in Shreve II.)

2. **The Lookback Functional.**

   (a) Let $h(x)$ be a functional that transforms a realization $(x_t)_{0 \leq t \leq T}$ into a real value $h(x)$. Show that if there is a constant $c$ so that for all $x$ and $y$
   
   $$|h(x) - h(y)| \leq c \sup_{0 \leq t \leq T} |x_t - y_t|,$$
   
   then $h$ is continuous in the sense of p. 25 of the notes.

   (b) Deduce that $h(x) = \sup_{0 \leq t \leq T} x_t$ is a continuous functional. [If this is too hard, solve the problem for $h(x) = \frac{1}{T} \int_0^T x_t dt$ instead.]

   (c) Use the result in (b) along with the Central Limit Theorem to simulate (in R or Splus) the value of $\max_{0 \leq t \leq 1} W_t$, where $W_t$ is an (additive) Brownian motion. You should use a binomial tree and keep in mind that the increments should be additive.

   (d) Compare the resulting histogram with that of the random variable $|N(0, 1)|$ (which can be simulated using the R-function “rnorm”). Write down any conjectures you may have about the distribution of $\max_{0 \leq t \leq 1} W_t$.

3. **The Asian Option.** For simplicity, assume that the interest rate $r = 0$, and let the stock price be given as a multiplicative Brownian motion with volatility $\sigma$, as on p. 23 of the notes. Assume that $S_0 = 100$ and that $\sigma = 0.20$, and consider an Asian call option with strike price $K = 105$ and maturity $T = 1$ year, i.e., with payoff

   $$\left( \int_0^T S_t dt - K \right)^+. $$

   Simulate a binomial tree in R (or possibly several trees for different values of $n$) to determine the price of this option. Justify the use of limit theorems (the central limit theorem, dominated convergence, etc).

   Sketch how the problem can be solved when $r \neq 0$, without changing the binomial tree that you used.