ONE PERIOD MODELS

\[ t = \text{TIME} = 0 \text{ or } 1 \]

BASIC INSTRUMENTS:

* \( S_t \): STOCK
* \( B_t \): RISKLESS BOND

\[ B_0 = 1 \quad B_1 = 1 + r \text{ or } e^r \]

* FORWARD CONTRACT:
AGREEMENT TO SWAP $$ FOR STOCK

- AGREEMENT TIME: \( t = 0 \)
- AGREEMENT PRICE: \( F_0 \)
- SWAP TIME \( t = 1 \)
- SWAP: $$ \( F_0 \) FOR 1 STOCK \( S_1 \)

ACTUAL INSTRUMENT: \( W_t \):

\[ W_0 = 0 \quad W_1 = S_1 - F_0 \]
ABITRAGE ARGUMENT: \( F_0 = e^r S_0 \)

If \( F_0 < e^r S_0 \):

FORM PORTFOLIO AT \( t = 0 \): NET POSITION

SELL 1 STOCK
BUY \( S_0 \# \) BONDS
ENTER 1 FORWARD CONTR

TOTAL PORTFOLIO

\[
V_t = -S_t + S_0 B_t + W_t
\]

VALUE:

\[
V_0 = -S_0 + S_0 + 0 = 0
\]

\[
V_1 = -S_1 + e^r S_0 + S_1 - F_0 = e^r S_0 - F_0 > 0
\]

\( V_t \) IS AN ARBITRAGE: MONEY FOR NOTHING
NOT SUPPOSED TO OCCUR
MORE GENERALLY...

* $K + 1$ ASSETS, PRICE $S^j_t$, $j = 1, \ldots, K$
  $S^0 = B$ IS BOND: $S^0_0 = 1$, $S^0_1 = e^r$

* PORTFOLIO $\Delta = (\Delta_1, \ldots, \Delta_K)$:

  HOLD $\Delta_j$ # OF SECURITY $S^j_t$

* VALUE OF PORTFOLIO:

  $$V_t(\Delta) = \sum_{j=0}^{K} \Delta_j S^j_t$$

* ARBITRAGE: A PORTFOLIO FOR WHICH

  $$V_0(\Delta) \leq 0$$
  $$V_1(\Delta) \geq 0$$
  $$V_1(\Delta) > 0$$ IN SOME SCENARIO

  ($= \text{ WITH PROBABILITY} > 0$)
TWO STATE MODEL FOR STOCK

\[ S_0 \xrightarrow{u} S_1 = uS_0 \quad \text{SCENARIO } H: \ S_1(H) \]
\[ S_0 \xrightarrow{d} S_1 = dS_0 \quad \text{SCENARIO } T: \ S_1(T) \]

Conditions to avoid arbitrage

Consider portfolio: buy \( \frac{1}{B_0} \) # of bonds, \( \frac{1}{S_0} \) # of stocks at time 0. Properties:

\[
V_0 = \frac{1}{B_0} B_0 - \frac{1}{S_0} S_0 = 0
\]
\[
V_1 = \frac{1}{B_0} B_1 - \frac{1}{S_0} S_1 = \begin{cases} e^r - u & \text{under scenario } H \\ e^r - d & \text{under scenario } T \end{cases}
\]

If \( e^r \geq u \):
\[
V_1 \geq 0 \quad \text{under } H \quad \text{ARBITRAGE}
\]
\[
V_1 > 0 \quad \text{under } T \quad \text{NOT ALLOWED}
\]

It follows that \( u > e^r \) (unless \( P(T)=0 \): \( e^r = 0 \) OK)

Similarly: portfolio \( \frac{1}{B_0} \) bonds, \( \frac{1}{S_0} \) stocks \( \Rightarrow e^r > d \)

CONCLUSION: no arbitrage implies \( u > e^r > d \).
CALL OPTIONS

\[ V_1 = (S_1 - K)^+ \quad V_0 = ??? \]

\[ S_0 \begin{cases} S_1 = uS_0 & \text{SCENARIO } H \\ S_1 = dS_0 & \text{SCENARIO } T \end{cases} \]

Suppose \( uS_0 > K > dS_0 \)
otherwise \( V_1 = S_1 - K \) (\( dS_0 \geq K \)) or \( V_1 = 0 \) (\( uS_0 \leq K \))

REPLICATING PORTFOLIO
Buy \( \Delta_0 \) bonds and \( \Delta_1 \) stocks at time 0

portfolio: \( V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t \)
replication: \( (S_1 - K)^+ = \Delta_1 B_1 + \Delta_2 S_1 \)

FINDING THE \( \Delta \)s: 2 EQUATIONS, 2 UNKNOWNS:

\[ uS_0 - K = \Delta_1 e^r + \Delta_2 uS_0 \] \( \text{SCENARIO } H \)
\[ 0 = \Delta_1 e^r + \Delta_2 dS_0 \] \( \text{SCENARIO } T \)

OR: \( \Delta_2 = \frac{uS_0 - K}{uS_0 - dS_0} \) and \( \Delta_1 = -e^{-r} \Delta_2 dS_0 \)

PRICE FOR THIS OPTION:

\[ V_0 = \Delta_1 B_0 + \Delta_2 S_0 \]
\[ = \Delta_2 e^{-r} (-dS_0 + e^r S_0) \]
ARGUMENT DEPENDS ON

- bond, stock can be bought or sold in any quantity
- bond, stock can be short sold (in the case of bond: this means that borrowing rate is same as lending rate)
- no bid-ask spread
- binomial model

Binomial model is oversimplification
“Brownian motion” is close to binomial model
Increasingly realistic models as the course progresses

ARGUMENT DOES NOT DEPEND

- Assumption of no arbitrage, except that $u > e^r > d$
MORE GENERAL DERIVATIVE SECURITIES
IN THE ONE PERIOD BINOMIAL MODEL

payoff \( V_1(H) \) or \( V_1(T) \)
or \( V_1 = f(S_1) \)

where \( f(s) = \begin{cases} (s - K)^+ \text{ call option} \\ (K - s)^+ \text{ put option} \\ \text{etc} \end{cases} \)

REPLICATING PORTFOLIO
Buy \( \Delta_0 \) bonds and \( \Delta_1 \) stocks at time 0

portfolio: \( V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t \)
replication: \( f(S_1) = \Delta_1 B_1 + \Delta_2 S_1 \)

FINDING THE \( \Delta \)'s: 2 EQUATIONS, 2 unknowns:

\[
\begin{align*}
f(uS_0) &= \Delta_1 e^r + \Delta_2 uS_0 \text{ SCENARIO } H \\
f(dS_0) &= \Delta_1 e^r + \Delta_2 dS_0 \text{ SCENARIO } T
\end{align*}
\]

OR: \( \Delta_2 = \frac{f(uS_0) - f(dS_0)}{uS_0 - dS_0} \) and \( \Delta_1 = e^{-r} \frac{uf(dS_0) - df(uS_0)}{u - d} \)

PRICE FOR THIS OPTION:

\( V_0 = \Delta_1 B_0 + \Delta_2 S_0 \)
DISCOUNTING

Discounted stock: \( S^*_t = S_t/B_t \)
Discounted bond: \( B^*_t = B_t/B_t = 1 \)
Discounted portfolio value: \( V^*_t = V_t/B_t \)

NUMERAIRE INVARIANCE

Portfolio in original numeraire: \( V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t \)

Portfolio in discounted numeraire:

\[
V^*_t(\Delta) = \frac{\Delta_0 B_t + \Delta_1 S_t}{B_t}
= \Delta_0 B^*_t + \Delta_1 S^*_t
\]

The number \( \Delta_0, \Delta_1 \) of bonds, stocks is the same in original and discounted numeraire

Exit interest. This is often convenient

TWO EQUATIONS, TWO unknowns
ON DISCOUNTED SCALE

\[
V^*_1(H) = \Delta_1 + \Delta_2 u S^*_0 \text{ SCENARIO } H
\]
\[
V^*_1(T) = \Delta_1 + \Delta_2 d S^*_0 \text{ SCENARIO } T
\]
PROBABILISTIC INTERPRETATION

Let $\pi(H)$, $\pi(T)$ be two numbers

From $V_t^* = \Delta_0 B_t^* + \Delta_1 S_t^*$:

$$
\pi(H)V_1^*(H) + \pi(T)V_1^*(T) = \pi(H)(\Delta_0 B_1^*(H) + \Delta_1 S_1^*(H)) + \pi(T)(\Delta_0 B_1^*(T) + \Delta_1 S_1^*(T)) = \Delta_0(\pi(H)B_1^*(H) + \pi(T)B_1^*(T)) + \Delta_1(\pi(H)S_1^*(H) + \pi(T)S_1^*(T)) = \Delta_0 B_0^* + \Delta_1 S_0^*
$$

provided

$$
\begin{align*}
\pi(H)B_1^*(H) + \pi(T)B_1^*(T) &= B_0^* \quad (**)
\pi(H)S_1^*(H) + \pi(T)S_1^*(T) &= S_0^* \quad (***)
\end{align*}
$$

$B_t^* = 1$: (***) $\implies$ $\pi(H) + \pi(T) = 1$

$\pi$ is a probability measure, provided $\pi(H), \pi(T) \geq 1$

This is the case since, by solving (***)-(***):

$$
\pi(T) = \frac{u - e^r}{u - d} \quad \text{and} \quad \pi(H) = \frac{e^r - d}{u - d}
$$

If $E_\pi$ is expectation under $\pi$:

$$
(*) \iff E_\pi V_1^* = V_0^*
\text{ (***)} \iff E_\pi S_1^* = S_0^*
$$

$\pi$ is the "RISK NEUTRAL" PROBABILITY MEASURE
THE RISK NEUTRAL PROBABILITY DISTRIBUTION:

\[ \pi : \quad S_j^0 = e^{-r} E_\pi S_1^j = E_\pi S_1^{j*} \text{ for all } j \]

FUNDAMENTAL THEOREM OF ARBITRAGE PRICING:

THERE EXISTS A RISK NEUTRAL MEASURE IF AND ONLY IF ARBITRAGE DOES NOT OCCUR.

EVALUATING PRICES USING \( \pi \):

\[
V_0 = \sum_{j=1}^{K} \Delta_j S_0^j \\
= e^{-r} \sum_{j=1}^{K} \Delta_j E_\pi S_1^j \\
= e^{-r} E_\pi V_1
\]

BUT WHAT IS \( \pi \)?

BRUNO DE FINETTI (1937): FORESIGHT: ITS LOGICAL LAWS, ITS SUBJECTIVE SOURCES:

“PROBABILITY DOES NOT EXIST.”
HORSE RACING
(from Baxter and Rennie: Financial Calculus. An introduction to derivative pricing.)

TWO HORSES: \( H_1, H_2 \)

<table>
<thead>
<tr>
<th>ACTUAL CHANCE OF WINNING</th>
<th>BETS PLACED ON HORSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_1 ) 25%</td>
<td>$ 5000</td>
</tr>
<tr>
<td>( H_2 ) 75%</td>
<td>$10000</td>
</tr>
<tr>
<td>TOTAL FOR BOOKIE</td>
<td>$15000</td>
</tr>
</tbody>
</table>

PRICE OF BETS

ACTUAL PROBABILITIES: \( \pi_1 = \frac{1}{4} \pi_2 = \frac{3}{4} \)

BETTING $1 ON HORSE \( H_1 \) : WIN $4
BETTING $1 ON HORSE \( H_2 \) : WIN $\frac{4}{3}

OUTCOME FOR BOOKIE:

WINNING HORSE $\$
\( H_1 \) \quad 15000 - 4 \times 5000 = -5000
\( H_2 \) \quad 15000 - \frac{4}{3} \times 10000 = 1666

A RISKY BUSINESS
HORSE RACING:

RISK NEUTRAL PROBABILITY

ACTUAL CHANCE OF WINNING BETS PLACED ON HORSE

\[ H_1 \text{ IRRELEVANT} \quad \$5000 \]

\[ H_2 \quad \$10000 \]

PRICE OF BETS:

\[ \$1 \text{ ON } H_1 \text{ GIVES } \frac{1}{\pi_1} \$ \text{ IF WIN} \]

\[ \$1 \text{ ON } H_2 \text{ GIVES } \frac{1}{\pi_2} \$ \text{ IF WIN} \]

OUTCOME FOR BOOKIE:

\[
\begin{array}{ccc}
\text{WINNER} & \text{\$} & \\
H_1 & 15000 - \frac{1}{\pi_1} \times 5000 & \\
H_2 & 15000 - \frac{1}{\pi_2} \times 10000 & \\
\end{array}
\]

OUTCOME = 0 IF \(\bar{\pi}_1 = \frac{1}{3}, \bar{\pi}_2 = \frac{2}{3}\) A SAFE BUSINESS.

JUST LIKE SELLING OPTIONS...
COMPLETE MARKETS

EQUIVALENT:

1) ALL RANDOM VARIABLES $V_1$ CAN BE REPRESENTED

$$V_1 = \Delta_0 B_0 + \Delta_1 S_1^1 + \cdots + \Delta_K S_1^K$$

2) $\pi$ IS UNIQUE

IN CALL OPTION — TWO STATE MODEL: DID NOT NEED TO VERIFY EXISTENCE OF PORTFOLIO