1. Let

\[ T = \begin{bmatrix}
  a_1 & b_1 \\
  c_2 & \ddots & \ddots \\
  \ddots & \ddots & b_{n-1} \\
  \vdots & \vdots & \ddots & c_n \\
  b_n & \cdots & \cdots & a_n
\end{bmatrix}. \]

(a) Show that if \( b_i c_{i+1} > 0 \) for all \( i \), there exists a diagonal matrix \( D \) such that \( B = DTD^{-1} \) is a symmetric matrix. Give an algorithm for computing \( D \).

(b) Consider the difference equation

\[-u_{i-1} + 2u_i - u_{i+1} + \frac{\sigma h}{2} (u_{i+1} - u_{i-1}) = f_i\]

with \( u_0 = \alpha \) and \( u_{N+1} = \beta \). Develop the corresponding matrix as a tri-diagonal matrix.

(c) Give conditions under which this tri-diagonal matrix may be made symmetric.

(d) From the scaled matrix \( B = DTD^{-1} \) compute the eigenvalues of the Jacobi matrix analytically and show that Jacobi converges.

(e) Suppose we wish to solve \( Ax = b \).

Show that if there exists a diagonal matrix \( D \) such that \( B = DAD^{-1} \) is symmetric and positive definite, then the SOR method converges for the original problem.

(Note: This implies that if such a \( D \) exists, it is not necessary to construct it).

2. Consider the 2-dimensional Laplace’s equation on the domain \( 0 < x < 1, 0 < y < 1 \), with the following boundary conditions:

\[ u(0, y) = 0, \quad u(1, y) = y, \quad u(x, 0) = 0, \quad u(x, 1) = x^2. \]

(a) Apply SOR to solve this problem on an \((N+1) \times (N+1)\) grid with \( N = 23 \). Choose a starting vector \( b \) with random entries \( b_i \in (0,1) \), and iterate until the relative error is less than \( 10^{-5} \). Solve the problem for five values of \( \omega \) and find an approximate ‘optimal’ value \( \hat{\omega} \). (Hint: You may want to fit a polynomial to the obtained number of iterations as function of \( \omega \) and find its minimum). Give the optimal value of \( \omega \) analytically.

(b) Now apply CG method to these equations and plot convergence behavior.
3. Consider Laplace’s equation as described in Problem 2. We are interested in applying the block Jacobi to this problem.
   (a) By examining the eigenvalues of $B_{BJ}$, show that the method converges.
   (b) Apply block SOR to this problem, using the optimal SOR parameter. The optimal parameter is given by
   $$ \omega = \frac{2}{1 + \sqrt{1 - \|B_{BJ}\|^2}}. $$

4. Let $A$ be an $n \times n$ symmetric matrix that is singular. Consider the iteration
   $$ Mx^{(k+1)} = Nx^{(k)} + b. $$
   (a) Show that at least one eigenvalue of $B = M^{-1}N$ equals 1.
   (b) Describe how you could modify a convergent algorithm to obtain a solution to a singular problem.
   (c) Using Gauss-Seidel, compute a solution to the problem
   $$ Ax = b, $$
   where $A$ is an $n \times n$ matrix of the form
   $$ A = \begin{bmatrix}
   1 & -1 & 0 & \cdots & 0 & 0 \\
   -1 & 2 & -1 & 0 & \cdots & 0 \\
   0 & -1 & 2 & -1 & 0 & \cdots \\
   & & & \ddots & & \ddots \\
   0 & \cdots & 0 & -1 & 2 & -1 \\
   0 & 0 & \cdots & 0 & -1 & 1
   \end{bmatrix} $$
   and $b$ is any pre-specified vector. Show how to modify $b$ so that it is in the range of $A$. Describe your modification to obtain a solution.
   (d) Re-order $A$ using the red-black ordering. Show that $n/2$ eigenvalues are zero for the Gauss-Seidel operator.

5. Suppose we want to solve the linear least squares problem
   $$ \min \|b - Ax\|^2 $$
   where $A$ is $m \times n$ ($m > n$), $b$ is $m \times 1$ and $\text{rank}(A) = r \leq n$. Consider the iteration scheme
   $$ (A^\top A + \lambda W)x_{i+1} = \lambda Wx_i + A^\top b, $$
   where $\lambda > 0$ and $W$ is an $n \times n$ symmetric positive definite matrix.
   (a) Let $M = A^\top A + \lambda W$ and $N = \lambda W$. Construct the iteration matrix $B = M^{-1}N$. Show that $B$ is diagonalizable, $\rho(B) < 1$ if $A^\top A$ is non-singular.
   (b) Assume $x_0 = 0$. Show that the sequence $\{x_i\}_{i=1}^\infty$ converges to the minimum $W$-norm solution to the least squares problem (5.1), even when $r < n$. Recall that for a symmetric positive definite $W$, the $W$-norm of a vector $x$ is defined by
   $$ \|x\|_W := (x^\top Wx)^{1/2}. $$
   (Hint: First consider the case when $W = I$, and show that if $x_0 = 0$, then $x_i \to A^\top b$.)
   (c) Show that when $A^\top A$ is non-singular,
   $$ \|e_{i+1}\|_W \leq \|e_i\|_W, $$
   where $e_i = x - x_i$.
   (d) Give the rate of convergence for this method.
(e) Consider the iteration
\[ r_i = A^\top b - A^\top Ax_i, \]
\[ (A^\top A + \lambda W)z_i = r_i, \]
\[ x_{i+1} = x_i + z_i. \]

Show that this is equivalent to (5.2).

(f) Going back to (5.2), show how to implement this algorithm using the QR decomposition of
\[
\begin{bmatrix}
A \\
\sqrt{\lambda}F
\end{bmatrix}
\]

where \( W = F^\top F \) is a factorization of \( W \).