Spatial Epidemics: Critical Behavior

Regina Dolgoarshinnykh\textsuperscript{1} and Steve Lalley\textsuperscript{2}

\textsuperscript{1} Columbia University

\textsuperscript{2} University of Chicago

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Mean Field Models
- Stochastic Logistic (SIS) Model
- Reed-Frost (SIR) Model
- Branching Envelopes
- Critical Behavior

Spatial Epidemic Models
- Spatial SIS and SIR Models
- Branching Random Walks and Superprocess Limits
- Spatial Epidemic Models: Critical Scaling
- Spatial Extent of SuperBM
Epidemics: Basic Questions

▶ How long do they last?
▶ How far do they spread?
▶ How many people are infected?
Stochastic Logistic (SIS) Model

- Population Size $N < \infty$
- Individuals susceptible (S) or infected (I).
- Infecteds recover in time $1$, then immediately susceptible.
- Infecteds infect susceptibles with probability $p$. 
Stochastic Logistic (SIS) Model

- Population Size $N < \infty$
- Individuals susceptible (S) or infected (I).
- Infecteds recover in time 1, then *immediately susceptible*.
- Infecteds infect susceptibles with probability $p$.

Dynamics:

\[
S_{t+1} = N - I_{t+1},
\]
\[
I_{t+1} = Y_{t,1} + Y_{t,2} + \cdots + Y_{t,N};
\]
\[
Y_{t,j} \sim \text{Bernoulli-}(1 - p)^I_t
\]
SIS Model: Example

\[ \text{SIS} \iff \text{Oriented Percolation on } K^N \times \mathbb{Z}^+ \]
SIS Model: Example

SIS $\iff$ Oriented Percolation on $K_N \times \mathbb{Z}_+$
Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability $p$. 
Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability $p$.

Dynamics:

\[
S_{t+1} = S_t - I_{t+1},
\]
\[
R_{t+1} = R_t + I_t,
\]
\[
l_{t+1} = Y_{t,1} + Y_{t,2} + \cdots + Y_{t,S_t},
\]
\[
Y_{t,j} \sim \text{Bernoulli}-(1 - p)^l_t
\]
Recovered individuals are immune from future infection.
SIR Model: Example
Reed-Frost and Random Graphs

Reed-Frost model is equivalent to the Erdös-Renyi random graph model:

- Individuals $\leftrightarrow$ Vertices
- Infections $\leftrightarrow$ Edges
- Epidemic $\leftrightarrow$ Connected Components
Reed-Frost and Random Graphs
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Branching Envelope of an Epidemic

- Each epidemic has a branching envelope (GW process)
- Offspring distribution: Binomial-($N, p$)
- Epidemic is dominated by its branching envelope
- When $I_t \ll S_t$, infected set grows $\approx$ branching envelope
Branching envelope of SIS Epidemic: Construction

- Branching process contains red particles and blue particles
- Red particles represent infected individuals
- Each blue has $\xi \sim \text{Binomial-}(N, p)$ blue offspring
- Each red has $\xi \sim \text{Binomial-}(N, p)$ red offspring
- Red offspring choose labels randomly in $[N]$
- Multiple labels: all but one turn blue
Example: SIS Epidemic and its Branching Envelope

\[ N = 80000 \]
\[ I_0 = 200 \]
\[ p = 1/80000 \]
Feller’s Theorem

- $Z_t^N$: Galton-Watson processes
- Offspring Distribution $F$: mean 1, variance $\sigma^2 < \infty$.
- Initial Conditions: $Z_0^N = aN$. 
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$$\Rightarrow Z_{Nt}^N / N \xrightarrow{D} Y_t$$

Feller process:

$$Y_0 = a$$
$$dY_t = \sigma \sqrt{Y_t} \, dB_t$$
Feller’s Theorem

- \( Z_t^N \): Galton-Watson processes
- Offspring Distribution \( F \): mean 1, variance \( \sigma^2 < \infty \).
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\[
\implies Z_{Nt}^N / N \xrightarrow{D} Y_t
\]

- Feller process:

\[
Y_0 = a \\
dY_t = \sigma \sqrt{Y_t} dB_t
\]
Feller’s Theorem

- $Z^N_t$: Galton-Watson processes
- Offspring Distribution $F$: mean $1 - \frac{b}{N}$, variance $\sigma^2 < \infty$.
- Initial Conditions: $Z^N_0 = aN$.

\[ \Rightarrow \quad Z^N_{Nt}/N \xrightarrow{D} Y_t \]

- Feller process with drift:

\[ Y_0 = a \]
\[ dY_t = \sigma \sqrt{Y_t} dB_t - bY_t dt \]
Critical Behavior: SIS Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $t$: $I^N_t$
- Initial Condition: $I^N_0 \sim bN^\alpha$.

\[ \text{Theorem: } I^N_N t / N^\alpha = Y_t \Rightarrow Y_0 = b; \quad dY_t = \sqrt{Y_t} dB_t \text{ if } \alpha < 1/2 \]
\[ dY_t = \sqrt{Y_t} dB_t - Y^2_t dt \text{ if } \alpha = 1/2 \]

Note: When $\alpha = 1/2$ the initial condition $b = \infty$ is permitted, as $\infty$ is an entrance boundary for the limit diffusion.
Critical Behavior: SIS Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $t$: $I_t^N$
- Initial Condition: $I_0^N \sim bN^\alpha$.

**Theorem:** $I_{N^\alpha t}^N / N^\alpha \Rightarrow Y_t$ where

$$Y_0 = b;$$

$$dY_t = \sqrt{Y_t} \ dB_t \quad \text{if} \quad \alpha < 1/2$$

$$dY_t = \sqrt{Y_t} \ dB_t - Y_t^2 \ dt \quad \text{if} \quad \alpha = 1/2$$

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**Note:** When $\alpha = 1/2$ the initial condition $b = \infty$ is permitted, as $\infty$ is an entrance boundary for the limit diffusion.
Example: Entrance Boundary

\[ N = 80000 \]
\[ I_0 = 10000 \]
\[ p = 1/80000 \]
Critical Behavior: SIS Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $t$: $I_t^N$
- Initial Condition: $I_0^N \sim bN^\alpha$.

Corollary: If $\alpha = 1/2$ then

$$\sum_{t \geq 0} I_t^N / N \Rightarrow \tau(b)$$

where $\tau(b) =$ first passage time to zero of Ornstein-Uhlenbeck process started at $b$.

Proof: Time change.
Critical Heuristics: SIS Epidemics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
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- **Critical Threshold**: $\#$ collisions/generations $\approx O(1)$
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Critical SIS Epidemic:

$$E(\#\text{collisions in generation } t + 1) \approx l_t^2 / N$$
Critical Heuristics: SIS Epidemics

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Critical SIS Epidemic:

$E(\#\text{collisions in generation } t + 1) \approx I_t^2 / N$

so observable deviation from branching envelope when

$I_t \approx \sqrt{N}$
Critical Behavior: Reed-Frost (SIR) Epidemics

- Population size: \( N \to \infty \)
- # Infected in Generation \( t \): \( I_t^N \)
- # Recovered in Generation \( t \): \( R_t^N \)
- Initial Condition: \( I_0^N \sim bN^\alpha \)

**Theorem:**

\[
\begin{pmatrix}
N^{-\alpha} I_t^N \\
N^{-2\alpha} R_t^N
\end{pmatrix} \xrightarrow{D} \begin{pmatrix}
I(t) \\
R(t)
\end{pmatrix}
\]

The limit process satisfies \( I(0) = b \) and

\[
dR(t) = l(t) \, dt
\]

\[
dl(t) = +\sqrt{l(t)} \, dB_t \quad \text{if } \alpha < \frac{1}{3}
\]

\[
dl(t) = +\sqrt{l(t)} \, dB_t - l(t)R(t) \, dt \quad \text{if } \alpha = \frac{1}{3}
\]
Critical Behavior: Reed-Frost (SIR) Epidemics

- Population size: \( N \to \infty \)
- # Infected in Generation \( t \): \( I_t^N \)
- # Recovered in Generation \( t \): \( R_t^N \)
- Initial Condition: \( I_0^N \sim bN^\alpha \)

**Corollary:** If \( \alpha = 1/3 \) then

\[
R_{\infty}^N / N^{2/3} \to \tau(b)
\]

where \( \tau(b) = \text{first passage time of } B(t) + t^2/2 \) to \( b \).

(Martin-Lof; Aldous)
Critical Heuristics: SIR Epidemics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Offspring in branching envelope :: attempted infections.
- Collisions: Infections of immunes not allowed.
- Critical Threshold: # collisions/generations $\approx O(1)$

Critical SIR Epidemic:

\[ E(\text{# collisions in generation } t + 1) \approx I_t(N - S_t)/N \]

so observable deviation from branching envelope when $I_t \approx N^{1/3}$.
Spatial SIS and SIR Epidemics

- Villages $V_x$ at Sites $x \in \mathbb{Z}$
- Village Size: $N$
- Nearest Neighbor Disease Propagation
- SIR or SIS Rules Locally:
Spatial SIS and SIR Epidemics

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  - Infected individual will infect susceptible at same or neighboring site with probability $(3N)^{-1}$
Spatial SIS and SIR Epidemics

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Random Graph Formulation:

- SIR Epidemic $\iff$ Percolation on $\mathbb{Z} \times \mathbb{K}_N$
- SIS Epidemic $\iff$ Oriented Percolation on $\mathbb{Z}^2 \times \mathbb{K}_N$. 
Spatial SIS and SIR Epidemics

- Villages $V_x$ at Sites $x \in \mathbb{Z}$
- Village Size: $= N$
- Nearest Neighbor Disease Propagation
- SIR or SIS Rules Locally:
  - Infected individual will infect susceptible at same or neighboring site with probability $(3N)^{-1}$

Associated Measure-Valued Processes

$$X_t^M = X_t^{M,N} :$$ measure that puts mass $1/M$ at $x/\sqrt{M}$ for each particle at site $x$ at time $t$. 
Critical Spatial SIS Epidemic: Simulation

Village Size: 20224

Initial State: 2048 infected at 0

Infection Probability: $p = \frac{1}{20224}$

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Spatial Epidemics: Critical Behavior
Branching Envelope of a Spatial Epidemic

Nearest Neighbor Branching Random Walk:

- Particle at $x$ puts offspring at $x - 1, x, x + 1$
- #Offspring are independent Binomial $-(N, p)$

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Branching Envelope of a Spatial Epidemic

Nearest Neighbor Branching Random Walk:

- Particle at $x$ puts offspring at $x - 1, x, x + 1$
- #Offspring are independent Binomial $(N, p)$

Associated Measure-Valued Processes

$X_t^M$: measure that puts mass $1/M$ at $x/\sqrt{M}$ for each particle at site $x$ at time $t$. 
Watanabe’s Theorem I

Let \( X_t^M \) be the measure-valued process associated to a critical nearest neighbor branching random walk. If

\[
X_0^M \Longrightarrow X_0
\]

then

\[
X_{Mt}^M \Longrightarrow X_t
\]

where \( X_t \) is the Dawson-Watanabe process (superBM). The DW process is a measure-valued diffusion.
Watanabe’s Theorem I

Let $X_t^M$ be the measure-valued process associated to a critical nearest neighbor branching random walk. If

$$X_0^M \Rightarrow X_0$$

then

$$X_{Mt}^M \Rightarrow X_t$$

where $X_t$ is the **Dawson-Watanabe process** (superBM). The DW process is a **measure-valued diffusion**.

**Note 1:** The total mass $\|X_t\|$ is a Feller diffusion.

**Note 2:** Watanabe is the spatial analogue of Feller’s theorem.
Watanabe’s Theorem II

Let $X_t^M$ be the measure-valued process associated to a critical nearest neighbor branching random walk with particles killed at rate $a/M$. If

$$X_0^M \overset{P}{\Rightarrow} X_0$$

then

$$X_{Mt}^M \overset{P}{\Rightarrow} X_t$$

where $X_t$ is the Dawson-Watanabe process with killing rate $a$. 

Dawson-Watanabe Process in 1D

**Superposition Principle:** Let $X_t^\mu$ be a Dawson-Watanabe process with killing rate $a$ and initial state

$$X_0^\mu = \mu.$$  

If $X^\mu$ and $X^\nu$ are independent Dawson-Watanabe processes with initial conditions $\mu$ and $\nu$ then

$$X_t^\mu \cup X_t^\nu \overset{D}= X_t^{\mu+\nu}$$
Dawson-Watanabe Process in 1D

**Superposition Principle:** Let $X_t^\mu$ be a Dawson-Watanabe process with killing rate $a$ and initial state

$$X_0^\mu = \mu.$$ 

If $X^\mu$ and $X^\nu$ are independent Dawson-Watanabe processes with initial conditions $\mu$ and $\nu$ then

$$X_t^\mu \cup X_t^\nu \overset{D}{=} X_t^{\mu+\nu}$$

**Absolute Continuity:** With probability 1, $X_t$ has a continuous density $X(t, x)$ relative to Lebesgue, and $X(t, x)$ is jointly continuous in $t, x$. (Konno-Shiga)
Scaling Limits: SIS Spatial Epidemics

**Theorem:** Let $X_t^N = X_t^{M,N}$ be the measure-valued process associated with critical SIS spatial epidemic with village size $N$ and scaling $M = N^\alpha$. If $X_0^N \Rightarrow X_0$ then

$$X_{Mt}^N \Rightarrow X_t$$

where

- If $\alpha < 2/3$ then $X_t$ is the Dawson-Watanabe process.
- If $\alpha = 2/3$ then $X_t$ is the Dawson-Watanabe process with location-dependent killing rate $X(t, x)^2$.

**Note:** $X_t^N$ is the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site $x$. 

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Scaling Limits: SIR Spatial Epidemics

Theorem: Let $X_t^N = X_t^{M,N}$ be the measure-valued process associated with critical SIR spatial epidemic with village size $N$ and scaling $M = N^\alpha$. If $X_0^N \Rightarrow X_0$ then

$$X_{Mt}^N \Rightarrow X_t$$

where

- If $\alpha < 2/5$ then $X_t$ is the Dawson-Watanabe process.
- If $\alpha = 2/5$ then $X_t$ is the Dawson-Watanabe process with killing rate

$$X(t, x) \int_0^t X(s, x) \, ds$$
Critical Scaling: Heuristics (SIS Epidemics)

- # Infected Per Generation: \( \approx M \)
- Duration: \( \approx M \) generations.
- # Infected Per Site: \( \approx \sqrt{M} \)
- # Collisions Per Site: \( \approx M/N \)
- # Collisions Per Generation: \( \approx M^{3/2}/N \)

So if \( M \approx N^{2/3} \) then # Collisions Per Generation \( \approx 1 \).
Critical Scaling: Heuristics (SIR Epidemics)

- # Infected Per Generation: $\approx M$
- Duration: $\approx M$ generations.
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Collisions Per Site: $\approx M^2/N$
- # Collisions Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Collisions Per Generation $\approx 1$. 

But how do we know that the infected individuals in generation $n$ don’t “clump”?

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Spatial Epidemics: Critical Behavior
Critical Scaling: Heuristics (SIR Epidemics)

- # Infected Per Generation: $\approx M$
- Duration: $\approx M$ generations.
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Collisions Per Site: $\approx M^2/N$
- # Collisions Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Collisions Per Generation $\approx 1$.

But how do we know that the infected individuals in generation $n$ don’t “clump”?
Watanabe’s Theorem Revisited

**Theorem:** Let $X^M(t, x)$ be the (rescaled) density process associated to a critical nearest neighbor branching random walk with particles killed at rate $a/M$. If

$$X_0^M \quad \Longrightarrow \quad X_0 \quad \text{in} \quad C(\mathbb{R})$$

then

$$X^M(Mt, x) \quad \Longrightarrow \quad X(t, x) \quad \text{in} \quad C(\mathbb{R})$$

where $X(t, x)$ is the the DW density process with killing rate $a$. 
Spatial Extent of Dawson-Watanabe Process

- \( X_t \) = Dawson-Watanabe process
- \( \mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t) \)
- \( u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x) \)
Spatial Extent of Dawson-Watanabe Process

- $X_t = \text{Dawson-Watanabe process}$
- $\mathcal{R}(X) := \cup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**Theorem (Dynkin):** For any finite interval $D$, $u_D(x)$ is the maximal nonnegative solution in $D$ of the differential equation

$$u'' = u^2$$
Spatial Extent of Dawson-Watanabe Process

- $X_t =$ Dawson-Watanabe process
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**Solution:** Weierstrass $\mathcal{P}$--Function

$$u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x - \omega)^2} - \frac{1}{\omega^2} \right\}$$

where the period lattice $L$ is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

$$C = \sqrt{6}a$$
Spatial Extent of Dawson-Watanabe Process

- $X_t = \text{Dawson-Watanabe process}$
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**General Initial Conditions:** For any finite Borel measure $\mu$ with support $\subset D$,

$$
-\log P(\mathcal{R}(X) \subset D \mid X_0 = \mu) = \int u_D(x) \mu(dx)
= \int \mathcal{P}_L(x/\sqrt{6}) \mu(dx)
$$

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