MathFinance 345/Stat390
Homework 7
Due November 21

Problem 1: Consider a contingent claim that pays \( S^n_T \) at time \( T \), where \( n \) is a positive integer.

(A) Show that the value of this contingent claim at time \( t \leq T \) is

\[ h(t, T)S^n_t \]

for some function \( h \) of \((t, T)\). **Hint:** Use the fact that the process \( S_t \) is a geometric Brownian motion. You should not need the Itô formula.

(B) Derive an ordinary differential equation for \( h(t, T) \) in the variable \( t \), and solve it. **Hint:** Your ordinary differential equation should be first-order, and it should involve only the short rate \( r_t \).

In Problems 2 and 3, let \( C(S_t, t) = C(S_t, t; K, T) \) be the price at time \( t \) of a European call option on the tradable asset \( (S_t) \) with strike price \( K \) and exercise time \( T \). Assume that the riskless rate of return \( r \) is constant and nonnegative, and that the share price process \( S_t \) of the underlying asset \( \text{STOCK} \) follows the stochastic differential equation

\[ dS_t = rS_t dt + \sigma S_t dW_t. \]

Problem 2:

(A) Show that the price function \( C \) satisfies the following symmetry properties: for any positive constant \( a \),

\[
\begin{align*}
(1) & \quad C(S, t; K, T) = C(S, 0; K, T - t) \\
(2) & \quad C(aS, t; aK, T) = aC(S, t; K, T) \quad \forall a > 0.
\end{align*}
\]

(B) Use the result of part (A) to derive an identity relating the partial derivatives \( C_S \) and \( C_K \).

(C) Find a PDE in the variables \( K, T \) for the function \( C(x, 0; K, T) \). (The equation should involve first and second partial derivatives.)

Problem 3. Denote by

\[ C^*(S_t, t) = e^{-rt}C(S_t, t) \]

the *discounted* value of the option.
(A) Show that, for each fixed $t \leq T$,
$$\lim_{x \to \infty} (C(x, t) - e^{-r(T+t)}x) = -K.$$  

(B) Show that, for each fixed $x > 0$, the function $C^*(x, t)$ is decreasing in $t$ and converges to $e^{-rT}(x - K)_+$ as $t \to T$.

(C) Show that $0 \leq C_x(x, t) \leq 1$ for all $x > 0$ and all $0 \leq t \leq T$. Also, verify that
$$\lim_{t \to T} C_x(x, t) = 1 \quad \text{if } x > K \quad \text{and}$$
$$\lim_{t \to T} C_x(x, t) = 0 \quad \text{if } x < K.$$  

Discuss the implications for hedging.

Note: In problems 2 and 3, you avoid the use of the Black–Scholes formula wherever possible. Instead, use the representation of the call option price as a (conditional) expectation.