1. **Double or Nothing.** The Optional Stopping Formula is not generally true if \(\tau \wedge n\) is replaced by \(\tau\). Following is a simple example: Let \(\xi_1, \xi_2, \ldots\) be an infinite sequence of independent Bernoulli random variables with success parameter \(\frac{1}{2}\): that is, \(P\{\xi_i = 1\} = P\{\xi_i = 0\} = \frac{1}{2}\). Define a sequence \((Z_n)_{n \geq 0}\) of random variables inductively as follows:

\[
\begin{align*}
Z_0 &= 1 \\
Z_n &= 2\xi_n Z_{n-1} & \forall n \geq 1.
\end{align*}
\]

(a) Prove that the sequence \((Z_n)_{n \geq 0}\) is a martingale relative to the usual filtration.

(b) Define \(\tau = \min\{n : Z_n = 0\}\). Prove that \(P\{\tau < \infty\} = 1\), and that \(\tau\) is a stopping time.

(c) Show that \(EZ_0 \neq EZ_\tau\).

2. **Arbitrages in Infinite Period Markets.** The economic assumption that efficient markets should not allow the existence of arbitrages may be reasonable in markets with finitely many trading periods, but not in markets with *infinitely* many trading periods. For example: Consider a homogeneous binary market with a risky asset \(\text{Stock}\) and a riskless asset \(\text{Bond}\) with rate of return \(r = 0\). The share price \(S_t\) of the asset \(\text{Stock}\) evolves as in the \(T\)-period homogeneous binary market: thus, there are constants \(d < 1 < u\) such that, given any partial scenario \(\omega = \omega_1 \omega_2 \ldots \omega_t\) of length \(t\),

\[
\begin{align*}
S_{t+1}(\omega_1 \omega_2 \ldots \omega_t +) &= uS_t(\omega_1 \omega_2 \ldots \omega_t) \\
S_{t+1}(\omega_1 \omega_2 \ldots \omega_t -) &= dS_t(\omega_1 \omega_2 \ldots \omega_t)
\end{align*}
\]

For the sake of simplicity, assume that

\[
\frac{u - 1}{u - d} = \frac{1 - d}{u - d} = \frac{1}{2}.
\]

Show that this market permits an arbitrage. (HINT: Construct a self-financing portfolio \(\theta\) whose value \(V_t^\theta\) evolves as the double-or-nothing martingale \(Z_n\) in problem 3. You will, of course, have to give a definition of an “arbitrage” in an infinite period market; your definition must be reasonable.)

3. **American Put Option:** The American put with strike \(K\) is a contract that gives the owner the right to *sell* one share of \(\text{Stock}\) for \(\$K\) at *any* time \(t = 1, 2, \ldots, T\). Assume that there is a riskless asset \(\text{Bond}\) whose rate of return is \(r > 0\). Give an example to show that, in certain markets and in certain circumstances, it may be better to exercise the put option early.

HINT: Consider a homogeneous 2-period binary market. Show that for some values of \(u, d, r\) and \(K\), you get a higher expected payoff by exercising the put at \(t = 1\) when the partial scenario is \(\_\) than waiting until \(t = 2\). Note that if \(r = 0\), then there is no advantage to early exercise.
4. First-Passage Time Distribution: Let $\xi_1, \xi_2, \ldots$ be an infinite sequence of independent Bernoulli-$\frac{1}{2}$ random variables, and let $S_n = \sum_{i=1}^{n} \xi_i$. Define $\tau = \min\{n : S_n = 1\}$ to be the first time that the “random walk” reaches the level 1. (Such random variables play an important role in barrier options, about which we shall have more to say later.) The purpose of this exercise is to find the distribution of the random variable $\tau$.

(a) Fix $z > 0$. Show that the sequence of random variables

$$Y_n = z^{S_n} / \varphi(z)^n$$

is a martingale relative to the natural filtration, where $\varphi(z) = (z + z^{-1})/2$.

(b) Show that if $z \geq 1$ then $E(1/\varphi(z)^\tau) = 1/z$. What goes wrong if $z < 1$?

(c) Conclude that for any $0 < \zeta < 1$, $E\zeta^\tau = \zeta P\{\tau = n\}$. What goes wrong if $\zeta > 1$?

(d) Use the fact that $E\zeta^\tau = \sum_{n=1}^{\infty} \zeta^n P\{\tau = n\}$ and calculus to find a formula for $P\{\tau = n\}$.

Reminder: The midterm exam will be held on October 24 during the regular class period in Kent 107. You may bring one page of notes.