EXERCISE SET 3: GRAPH COLORINGS

The Potts model is a generalization of the antiferromagnetic Ising model in which the set of spins \{\pm 1\} is replaced by a finite set \([k]\) of \(k\) colors. The energy \(H(x)\) of a configuration \(x\) of colors on the vertices of a finite graph \(G\) is defined to be the number of edges that connect vertices of the same color. The Gibbs state \(\mu_\beta\) at \(\beta = 1/(kT)\) is the probability distribution on color configurations that assigns to a given configuration \(x\) probability proportional to \(\exp\{-\beta H(x)\}\). The zero-temperature limit is therefore the uniform distribution on the set of all color configurations in which no two adjacent vertices are assigned the same color. Call such configurations admissible \(k\)-colorings, or simply \(k\)-colorings. The following problems address some issues in the enumeration of admissible \(k\)-colorings.

1. Define \(R_{m,n}\) to be the \(m \times n\) rectangular lattice with free boundary conditions, that is, the graph with vertex set \([m] \times [n]\) and edges connecting those pairs of vertices that differ by either \((\pm 1, 0)\) or \((0, \pm 1)\). Define \(U_k(m, n)\) to be the number of admissible \(k\)-colorings of the graph \(R_{m,n}\).
   (a) Find an explicit formula for \(U_3(m, n)\).
   (b) To the extent that the relevant matrix calculations are possible, generalize to \(k > 3\).
   HINT: Build admissible \(3\)-colorings column by column, starting at the leftmost column. For each column, you will have to determine the number of ways to adjoin a new column in such a way that the configuration remains admissible.

2. Define \(C_{m,n}\) to be the \(m \times n\) rectangular lattice with cylindrical boundary conditions. This is the graph with vertex set \([m] \times [n]\) and edges connecting those pairs \((x, y), (x', y')\) such that either
   \[x = x', \quad |y - y'| = 1 \quad \text{or} \quad y = y', \quad x - x' = \pm 1 \mod m.\]
   Define \(V_k(m, n)\) to be the number of admissible \(k\)-colorings of \(C_{m,n}\).
   (a) Find an explicit formula for \(V_3(m, n)\).
   (b) Prove that the limits
   \[
   \lim_{n \to \infty} n^{-2} \log V_3(n, n) \quad \text{and} \quad \lim_{n \to \infty} n^{-2} \log U_3(n, n)
   \]
   exist. Are they equal?

3. What? Haven’t had enough yet? Then try Exercise 37 (b), Chapter 4, in R. Stanley, Enumerative Combinatorics I: Define \(T_{m,n}\) to be the \(m \times n\) rectangular lattice with toroidal boundary conditions. This is the graph with vertex set \([m] \times [n]\) and edges connecting those pairs \((x, y), (x', y')\) such that either
   \[x = x', \quad y = y', \quad \text{and} \quad y - y' = \pm 1 \mod n \quad \text{or} \quad x - x' = \pm 1 \mod m.\]
   Define \(W_k(m, n)\) to be the number of admissible \(k\)-colorings of \(C_{m,n}\).
   (a) Prove that \(\log W_3(n, n) = (3n^2/2) \log(4/3) + o(n^2)\).
   (b) Prove that \(\log W_3(n, n) = (3n^2/2) \log(4/3) - \pi/6 + o(1)\).

Stanley rates (a) at level 4-, and (b) at level 5.