EXERCISE SET 2: STRASSEN’S MONOTONE COUPLING
THEOREM

Theorem 1. (Strassen) Let \((X, \leq)\) be a finite poset, and let \(\mu, \nu\) be probability distributions on \(X\). If \(\mu \leq \nu\) then on some probability space (in fact, on any probability space supporting a random variable uniformly distributed on the unit interval) are defined \(X\)-valued random variables \(M, N\) with distributions \(\mu, \nu\), respectively, such that

\[
\text{(1)} \quad M \leq N.
\]

* means that I don’t know the answer.

** means that I don’t know the answer even after thinking about it for a while.

*** means that I really don’t know the answer, and that a solution may be worthy of publication.

1. In class we proved the special case of Strassen’s theorem where the probability distributions \(\mu\) and \(\nu\) assign rational probabilities \(k/n\), with common denominator \(n\), to subsets of \(X\). Deduce the general case from the special case. HINT: Enlarge the poset \(X\) to \(X^*\) by adjoining elements \(x_+\) and \(x_-\) that are greater and smaller, respectively, than all other elements. For arbitrary probability distributions \(\mu, \nu\) on \(X\) such that \(\mu \leq \nu\), construct probability distributions \(\mu^*_n \leq \nu^*_n\) on \(X^*\) that assign only probabilities \(k/n\), in such a way that \(\mu^*_n \to \mu\) and \(\nu^*_n \to \nu\) as \(n \to \infty\).

2. (Optional\(^1\)). Learn the MAX-FLOW MIN-CUT Theorem\(^2\), and show that both HALL’S theorem and STRASSEN’S theorem may be deduced from it.

3. Let \(\mu\) and \(\nu\) be probability distributions on (the Borel sets of) the infinite product space \(\mathcal{X} = \{0,1\}^\mathbb{N}\) such that \(\mu \leq \nu\) with respect to the usual partial order. Prove that the conclusion of Strassen’s theorem holds.

4*. Does Strassen’s theorem extend to every infinite poset?

5. Let \((\mathcal{X}, \leq)\) be a finite poset, and let \(\mu_1, \mu_2, \ldots, \mu_n\) be probability distributions on \(\mathcal{X}\) such that

\[
\text{(2)} \quad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.
\]

\(^1\)Of course, in this class, everything is optional

\(^2\)See, for instance, the book Graph Theory by B. Bollobas
Prove that, on some probability space are defined $X$–valued random variables
\[(3) \quad M_1 \leq M_2 \leq \cdots \leq M_n\]
with marginal distributions $\mu_1, \mu_2, \ldots, \mu_n$, respectively.

6**. Let $(\mathcal{X}, \leq)$ and $(\mathcal{Y}, \leq)$ definite posets, and let $\{\mu_x\}_{x \in \mathcal{X}}$ be probability distributions on $\mathcal{Y}$ such that if $x \leq x_*$ then
\[\mu_x \leq \mu_{x_*}.\]
Prove that on some probability space there exist $\mathcal{Y}$–valued random variables $Y_x$, for all $x \in \mathcal{X}$, such that for each $x$ the marginal distribution of $Y_x$ is $\mu_x$, and
\[x \leq x_* \implies Y_x \leq Y_{x_*}.\]