Problem 3: Last exit decompositions. Let $T$ be an ergodic, measure-preserving transformation of a probability space $(\Omega, F, P)$ and let $\xi$ be an integer-valued, integrable random variable on $(\Omega, F, P)$ such that $0 < E\xi = \mu < \infty$. Let $S_0 = 0$ and $S_{n+1} = S_n + \xi_{n+1}$ where $\xi_{n+1} = \xi \circ T^n$. Let $F$ (for final) be the event

$$F = \{ \omega : S_n(\omega) \geq 1 \text{ for all } n \geq 1 \},$$

and for each $m \geq 1$ let

$$\tau_0(\omega) = 0 \quad \text{and} \quad \tau_m(\omega) = \inf\{n > \tau_{m-1}(\omega) : T^n(\omega) \in F\}$$

with the convention that $\tau_m = \infty$ if either $\tau_{m-1} = \infty$ or if there is no $n > \tau_{m-1}$ such that $T^n(\omega) \in F$.

(A) Show that $P(F) > 0$.

Solution: Birkhoff’s ergodic theorem implies that $S_n/n \to \mu > 0$ almost surely, and so $S_n \to \infty$ almost surely. Consequently, with probability one, there exists some (random) time $L \geq 0$ such that the random walk $S_{n \geq 0}$ visits the negative halfline $(-\infty, 0]$ for the last time at time $n = L$. At this time $L$, it must be the case that

$$T^L(\omega)(\omega) \in F.$$

Since $L < \infty$ with probability one, there must exist some $k \geq 0$ such that $L = k$ with positive probability, and so for this $k$, $P(\omega : T^k(\omega) \in F) > 0$. But the transformation $T$ is measure-preserving, so $P(\omega : T^k(\omega) \in F) = P(F)$. Hence, $P(F) > 0$.

(B) Show that $\tau_m < \infty$ almost surely.

Solution: Birkhoff’s theorem implies that with probability one, $n^{-1} \sum_{i=1}^n 1_F \circ T^i \to P(F)$. Since $P(F) > 0$ (by part (A)), it follows that with probability one, $T^i \omega \in F$ for infinitely many $i \geq 1$. This clearly implies that $\tau_m < \infty$ almost surely.

(C) Show that $E \tau_m < \infty$.

Solution: This statement is FALSE (my bad!). A counterexample can be constructed using a two-sided stationary renewal process whose inter-arrival time distribution has finite mean but infinite variance. For instance, let $\{Y_n\}_{n \in \mathbb{Z}}$ be independent, identically distributed with distribution

$$P\{Y_i = k\} = \frac{1}{\zeta(3)k^3} = f_k \quad \text{for } k = 1, 2, \ldots$$
(recall that $\zeta(s) = \sum_{k \geq 1} k^{-s}$), and let $\bar{Y}$ be independent of the sequence $\{Y_n\}_{n \in \mathbb{Z}}$ with the size-biased distribution distribution

$$P\{\bar{Y} = k\} = kf_k \zeta(3)/\zeta(2).$$

Conditional on $\bar{Y}$, let $L$ be uniformly distributed on the set of integers $\{0, 1, 2, ..., \bar{Y} - 1\}$: more precisely, let $U$ be a uniform-[0,1] random variable independent of the sequence $\{Y_n\}_{n \in \mathbb{Z}}$ and $\bar{Y}$, and set

$$L = \sum_{i=0}^{\bar{Y}-1} i \times 1\{[\bar{Y}U] = i\}$$

where $[\cdot]$ denotes greatest integer. Then the random variable $W_0 = \bar{Y} - L$ has the distribution

$$P\{W_0 = k\} = \mu^{-1} \sum_{j \geq k} f_j$$

where $\mu = \sum_{k \geq 1} kf_k = \zeta(2)/\zeta(3)$,

and $EL = \infty$. (You should check this.) You should recognize this distribution as the limiting residual lifetime distribution. In particular, if we now set

$$W_n = W_0 + \sum_{i=1}^{n} Y_i;$$

then the sequence $\{W_n\}_{n \geq 1}$ is a delayed renewal process with the property that its residual lifetime process is in steady-state. Finally, use the renewal process to define a stationary, ergodic sequence $\{\xi_m\}_{m \geq 0}$ by

$$\xi_m = Y_n + 1 \quad \text{if} \quad m = W_{n+1};$$

$$= -1 \quad \text{if} \quad \sum_{n \geq 0} 1\{m = W_n\} = 0,$$

and set

$$S_m = \sum_{i=1}^{m}.$$

You should observe that the value $S_m$ is the residual lifetime at time $m$ plus the number of renewals up to time $m$ (it might be helpful to draw the graph as a sample path). The occurrence times of the (delayed) renewal process $\{W_n\}_{n \geq 0}$ are precisely the times $\tau_m$ that $S_{\tau(m)+j} - S_{\tau(m)} \geq 1$ for all $j \geq 1$. Since $EW_0 = \infty$ (by construction), it follows that $E\tau(m) = \infty$. 