Problem 1. Let $\mathcal{Y}$ be a compact metric space and let $h : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ be a continuous function on $\mathcal{Y}$. Let $P$ be a Borel probability measure on $\mathcal{Y}$, and let $T : \mathcal{Y} \to \mathcal{Y}$ be an ergodic measure-preserving transformation. Prove that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} h(T^i(x), T^j(x))$$

exists for almost every $x \in \mathcal{Y}$, and evaluate the limit. Can you prove an analogous theorem for the case where $h$ is only assumed to be bounded and measurable?

Problem 2. Let $T_{\alpha} : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ be the transformation defined by $T_{\alpha}(x, y) = (x + \alpha, y + 2x + \alpha)$ where addition is done modulo 1. Assume that $\alpha$ is irrational.

(A) Show that $T_{\alpha}$ preserves Lebesgue measure.
(B) Show that there is no other invariant Borel probability measure for $T_{\alpha}$.

Problem 3. Multiparameter Ergodic Theorem Let $S$ and $T$ be ergodic, measure-preserving transformations of a probability space $(\Omega, \mathcal{F}, \mu)$, and let $Y : \Omega \to \mathbb{R}$ be a bounded random variable such that $EY = 0$. Using the strategy outlined below, prove that

$$\lim_{m, n \to \infty} \frac{1}{mn} \sum_{m=0}^{m} \sum_{j=0}^{n} Y \circ T^j \circ S^i = 0 \quad \text{almost surely.} \quad (1)$$

(A) Fix $\varepsilon > 0$ small. Use Birkhoff’s Theorem to show that for sufficiently large $m(\varepsilon)$,

$$E \sup_{m \geq m(\varepsilon)} \left| \frac{1}{m} \sum_{i=1}^{m} Y \circ S^i \right| < \varepsilon^2.$$

(B) Deduce from (A) that $\mu(B_\varepsilon) \leq \varepsilon$, where $B_\varepsilon$ is the event

$$B_\varepsilon = \left\{ \sup_{m \geq m(\varepsilon)} \left| \frac{1}{m} \sum_{i=1}^{m} Y \circ S^i \right| \geq \varepsilon \right\}.$$

(C) Use Birkhoff’s Theorem a second time to show that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} 1_{B_\varepsilon} \circ T^j \leq \varepsilon.$$

and use this to deduce \(1\).
**Site Percolation** The next problem concerns a process known as *site percolation*. Let \( \{X_{(i,j)}\}_{(i,j)\in \mathbb{Z}^2} \) be a stationary array of Bernoulli-\( p \) random variables, i.e., an array such that
\[
(X_{i+1,j})_{(i,j)\in \mathbb{Z}^2} \overset{\text{d}}{=} (X_{i,j})_{(i,j)\in \mathbb{Z}^2} \quad \text{and} \quad (X_{i,j+1})_{(i,j)\in \mathbb{Z}^2} \overset{\text{d}}{=} (X_{i,j})_{(i,j)\in \mathbb{Z}^2}.
\]
The array \( \{X_{(i,j)}\}_{(i,j)\in \mathbb{Z}^2} \) is said to be *ergodic* if both the horizontal and vertical induced shift mappings are ergodic. Let \( \kappa_{(i,j)} \) be the (random) set of all vertices \((m,n)\) of the lattice such that there is a path from \((i,j)\) to \((m,n)\) that passes only through vertices \((m,n)\) such that \(X_{(m,n)} = 1\). The set \( \kappa_{(i,j)} \) is called the *connected cluster* attached to \((i,j)\); note that it is empty if \(X_{(i,j)} = 0\). Note also that for any two vertices \((i,j)\) and \((m,n)\), the sets \( \kappa_{(i,j)} \) and \( \kappa_{(m,n)} \) are either equal or disjoint. Let
\[
\alpha(p) = P\{|\kappa_{(0,0)}| = \infty\};
\]
this is called the *percolation probability*.

**Problem 4.** Assume that the array \( \{X_{(i,j)}\}_{(i,j)\in \mathbb{Z}^2} \) is ergodic.

(A) Show that the number of distinct, infinite connected clusters is almost surely constant.

(B) Give examples to show that for any integer \( 0 \leq k \leq \infty \) there is a stationary, ergodic array \( \{X_{(i,j)}\}_{(i,j)\in \mathbb{Z}^2} \) for which the number of distinct, infinite connected clusters is almost surely \( k \).

(C) Show that if the random variables \( X_{(i,j)} \) are independent, identically distributed Bernoulli-\( p \), then the number of distinct, infinite connected clusters is either 0, 1, or \( \infty \). HINT: Show that if there is positive probability that there are 13 distinct, infinite connected clusters then there is positive probability that there is a unique infinite connected cluster.

**Note:** It is known that in fact the number of infinite connected cluster for i.i.d. Bernoulli site percolation is almost surely either 0 or 1, but this is harder to prove.