Problem 1. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a countably infinite orthonormal set in a Hilbert set \( H \). (a) Show that \( \{u_n\}_{n \in \mathbb{N}} \) is not compact, but is closed and bounded. Thus, the Heine-Borel theorem does not hold in infinite-dimensional Hilbert spaces. (b) Let \( \alpha_n \) be a sequence of scalars such that \( \sum_{n=1}^{\infty} \alpha_n^2 < \infty \), and define \( A \) to be the set of all \( x \in H \) with expansions \( x = \sum_{n=1}^{\infty} \beta_n u_n \), where \( |\beta_n| \leq \alpha_n \) for all \( n \). Show that \( A \) is compact.

Problem 2. A \( \sigma \)-algebra \( F \) on \( \Omega \) is said to be countably generated if there is a countable set \( \{F_n\}_{n \geq 1} \) of subsets of \( \Omega \) such that \( F \) is the smallest \( \sigma \)-algebra containing all of the sets \( F_n \).

(a) Prove that if \( F \) is countably generated then for any probability measure \( P \) on \( (\Omega, F) \) and every \( 1 \leq p < \infty \) the space \( L^p(\Omega, F, P) \) is separable. Hint: It is enough to show that there is a countable set \( \{B_n\} \subset F \) such that for every \( F \in F \) and every \( \varepsilon > 0 \) there is some \( B_m \) such that \( P(B_n \Delta F) < \varepsilon \).

(b) Prove that if \( F \) is countably generated then there is a real-valued, \( F \)-measurable random variable \( X \) such that \( F = \sigma(X) \). (Recall that \( \sigma(X) \) is the smallest \( \sigma \)-algebra with respect to which \( X \) is measurable; equivalently, it is the \( \sigma \)-algebra consisting of all events of the form \( \{X \in B\} \) where \( B \) is a Borel subset of \( \mathbb{R} \).) Hint: Let \( \{A_n\}_{n \in \mathbb{N}} \) be a countable set of generators for \( F \). Try to define \( X \) in such a way that the value of \( X \) simultaneously encodes the values of all of the indicators \( 1_{A_n} \).

Problem 2 implies that \( L^p([0,1], B_{[0,1]}, \text{Lebesgue}) \) is separable. In fact, it isn’t hard to show that the set of step functions with rational values is dense. There are, however, other interesting and useful dense sets. The next problem indicates how ideas from probability theory can be used to show that the polynomials are dense in \( L^p \). Since step functions can be arbitrarily well-approximated by continuous functions in \( L^p \) (why?), it is enough to show that every continuous function can be arbitrarily well-approximated by polynomials.

Problem 3. Let \( f : [0,1] \to \mathbb{R} \) be continuous. Show that for every \( p \in [0,1] \),
\[
f(p) = \lim_{m \to \infty} \sum_{k=0}^{m} f(k/m) \left( \frac{m}{k} \right)^p (1-p)^{m-k}
\]
and that the convergence is uniform in \( p \). This implies Weierstrass’ theorem that every continuous function can be uniformly approximated by polynomials. Hint: For the uniform convergence you will need the fact that every continuous function on \([0,1]\) is uniformly continuous.

Problem 4. Irrational Rotations: (a) Let \( f : \mathbb{R} \to \mathbb{C} \) be a continuous, periodic function with period 1. Prove that for any irrational real number \( \alpha \),
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} f(k\alpha) = \int_{0}^{1} f(t) \, dt
\]
HINT: First prove that the assertion is true for the complex exponential functions $e_{n}(t) := \exp\{2\pi i n \theta\}$.

(b) Denote by $\langle x \rangle$ the fractional part of $x \in \mathbb{R}$, i.e., $\langle x \rangle = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the integer part of $x$. Show that for any $\alpha \in [0, 1]$ the mapping $T_{\alpha} : [0, 1] \to [0, 1]$ defined by $T_{\alpha}(x) = \langle x + \alpha \rangle$ is measure-preserving (relative to Lebesgue measure on $[0, 1]$), and prove that $T_{\alpha}$ is ergodic if and only if $\alpha$ is irrational.

**Problem 5.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $T : \Omega \to \Omega$ be an invertible, measure-preserving transformation. A random variable $f \in L^{2}(\Omega, \mathcal{F}, P)$ is said to be a cocycle relative to $T$ if there is a random variable $g \in L^{2}(\Omega, \mathcal{F}, P)$ such that $f = g - g \circ T$.

(a) Prove that if $T$ is ergodic then the set of cocycles is dense in the Hilbert space $L^{2}_{0}(\Omega, \mathcal{F}, P)$ consisting of all $f \in L^{2}(\Omega, \mathcal{F}, P)$ with expectation 0. HINT: If not then there would be a nontrivial random variable $h \in L^{2}(\Omega, \mathcal{F}, P)$ that is orthogonal to every cocycle. Show that any such $h$ must be $T$-invariant.

(b) Deduce von Neumann’s Ergodic Theorem: If $T$ is an ergodic, measure-preserving transformation of a probability space $(\Omega, \mathcal{F}, P)$ then for every $f \in L^{2}(\Omega, \mathcal{F}, P)$,

$$\frac{1}{m} \sum_{k=0}^{m-1} f \circ T^{k} \overset{L^{2}}{\to} Ef.$$  

(c) Show that if $f$ is a cocycle then the central limit theorem fails for the partial sums $\sum_{k=1}^{m} f \circ T^{k}$; in particular, show that these partial sums have bounded second moments.

(d) The results of parts (a)–(b) hold in particular for the forward shift $T$ on the infinite product space $\Omega = \mathbb{R}^{\mathbb{Z}}$ with a product measure $P$, that is, a probability measure $P$ such that for any cylinder set $B_{-m} \times B_{-m+1} \times \ldots \times B_{n}$,

$$P(B_{-m} \times B_{-m+1} \times \ldots \times B_{n}) = \prod_{k=-m}^{n} \mu(B_{k}),$$

where $\mu$ is a Borel probability measure on $\mathbb{R}$. In this case the coordinate random variables $X_{k}$ are i.i.d. with common distribution $\mu$. Part (a) implies that if $\mu$ has finite second moment and mean 0 then $X_{0}$ can be arbitrarily well-approximated by a cocycle, but it doesn’t tell you how to find such approximators. Give an explicit formula for cocycles $g - g \circ T$ that converge to $X_{0}$.