Ergodic theory of rigid motions. In these problems, let \((Y, d)\) be a compact metric space and let \(T : Y \to Y\) be an isometry, that is, a continuous mapping such that for any two points \(x, y \in Y\),
\[
d(x, y) = d(Tx, Ty).
\]
(Example: Any rotation of the unit circle \(T^1\) is an isometry.) For any Borel measurable function \(f : Y \to \mathbb{R}\) and any integer \(n \geq 1\) let \(A_n f : Y \to \mathbb{R}\) be the \(n\)th sample average
\[
A_n f(y) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i y),
\]
where as usual \(T^i = T \circ T \circ \cdots \circ T\) is the \(i\)−fold iterate of \(T\).

**Problem 1.** Show that if \(f : Y \to \mathbb{R}\) is continuous then every subsequence \(A_{n_k} f\) of the sequence \(A_n f\) has a uniformly convergent subsequence. **Hint:** If you don’t remember it, look up the Arzela-Ascoli theorem for compact metric spaces in Wikipedia.

**Problem 2.** Show that if \(f : Y \to \mathbb{R}\) is continuous then the limit \(a(y) = \lim_{k \to \infty} A_{n_k} f(y)\) of any subsequence is continuous and \(T\)−invariant, i.e., \(a(Ty) = a(y)\) for every \(y \in Y\).

**Problem 3.** Show that there is only one subsequential limit in (A), in particular, if \(f : Y \to \mathbb{R}\) is continuous then there is a \(T\)−invariant, continuous function \(a_f : Y \to \mathbb{R}\) such that
\[
\lim_{n \to \infty} A_n f(y) := a_f(y) \quad \text{uniformly for } y \in Y.
\]

**Problem 4.** Assume now that \(T\) has a dense orbit, that is, there exists \(y \in Y\) such that the sequence \(\{T^i y\}_{i \geq 0}\) is dense in \(Y\). Prove that for every continuous \(f : Y \to \mathbb{R}\) the function \(a_f\) is constant.

**Problem 5.** Conclude that if \(T\) has a dense orbit then the mapping \(T : Y \to Y\) has at most one invariant Borel probability measure \(\mu\). (Such mappings are called uniquely ergodic.) **Hint:** Birkhoff’s Ergodic Theorem could be useful here.

**Note:** The Riesz Representation Theorem (cf., for example, W. Rudin, Real and Complex Analysis, ch. 2) implies that an invariant measure exists, as it can easily be checked that the functional \(f \mapsto a_f\) is a bounded, linear functional on \(C(Y)\).
**Problem 6.** Show that if \( \theta \) is *irrational*, then *every* orbit of \( T \) is dense in \( T^1 \). Use this to conclude that the uniform distribution \( \lambda \) (=Lebesgue /2\pi) is the unique invariant Borel probability measure.

**Problem 7.** Use the results of problems 3, 4, and 6 to conclude that for every Borel measurable function \( f : T^1 \to \mathbb{R} \) such that \( E|f| < \infty \),
\[
\lim_{n \to \infty} A_n f(x) = E_{\lambda} f \quad \text{almost surely(\( \lambda \)).}
\]
Conclude that the measure-preserving mapping \( T \) is *ergodic*.

*Hint:* Problems 3, 4, and 6 imply that this is true for every *continuous* \( f \) (why?). Use the \( L^1 \)—density of continuous functions and Wiener’s Maximal Inequality to show that it holds for all \( L^1 \) functions.

**Bonus Problem.** *(Not to be turned in unless you’re really gung-ho.)*

**Cut points.** Let \( \xi_1, \xi_2, \ldots \) be independent, identically distributed integer-valued random variables with finite first moment and positive mean \( \mu := E\xi_1 \), and let \( S_n = \sum_{i=1}^n \xi_i \) be the \( n \)th partial sum. For each \( x \geq 1 \) let \( T_x = \min\{ n : S_n \geq x \} \). Say that an integer \( y \geq 1 \) is a *cut point* for the random walk \( S_n \) if
\[
S_{T_y} = y \quad \text{and} \quad \min_{k \geq 1} S_{T_y + k} \geq y + 1.
\]

(A) Show that with probability one there are infinitely many cut points.

(B) Show that if \( Y_1 < Y_2 < Y_3 < \ldots \) are the cuts points arranged in increasing order, then with probability one
\[
\lim_{n \to \infty} Y_n / n = \alpha
\]
exists, and identify the limit \( \alpha \).

*Hint:* Use Birkhoff’s ergodic theorem. You might also find it helpful to show that, without loss of generality, the sequence of random variables \( \xi_1, \xi_2, \ldots \) can be extended to a doubly-infinite sequence \((\xi_i)_{i \in \mathbb{Z}}\) of independent, identically distributed random variables.