9. Identification of causal structures

$X_r$ set of random variables in $X_r$

$P$ probability distribution on $X_r$

$J(P)$ set of conditional independence relations which hold for $P$

**Question:**

Can we find a graphical representation for $J(P)$?

$G$ class of graphs $G=(V, E)$

**Aim:**

Find $G \in G$ such that $P$ satisfies the global Markov property w.r.t. $G$. 
**Definition (Consistency)**

A graph $G = (V, E)$ is consistent with $P$ if $P$ satisfies the global Markov property w.r.t. $G$,

$$J(G) \subseteq J(P).$$

If $G$ is complete $\Rightarrow J(G) = \emptyset$

$\Rightarrow G$ consistent with $P$

$G(P) = \{ G \in G | G$ consistent with $P \}$

**Definition (Minimality)**

A graph $G \in G(P)$ is minimal (for $P$) if for all graphs $G' \in G(P)$

$$J(G) \subseteq J(G') \Rightarrow G = G'$$
Recovering DAG structures

Recall: Two DAGs $G_1$ and $G_2$ are Markov equivalent,

$$J(G_1) = J(G_2),$$

if

- they have the same skeleton
  
  \[(a,b \text{ adjacent in } G_1) \iff (a,b \text{ adjacent in } G_2)\]

- they have the same immoralities
  
  \[(\text{subgraphs of the form } a \rightarrow c \leftarrow b)\]

**IC algorithm (Inductive causation)**

1. Find undirected graph $G^{(u)} = (V, E^{(u)})$ such that

   $$a \rightarrow b \in E^{(u)} \iff \exists S_{ab} \subseteq V \setminus \{a, b\}: X_a \perp X_b | X_{S_{ab}} [P]$$

2. Obtain partially directed graph $G^{(p)} = (V, E^{(p)})$ from $G^{(u)}$:

   $$a \rightarrow c \leftarrow b \text{ in } G^{(u)}, c \notin S_{ab} \Rightarrow a \rightarrow c \leftarrow b \text{ in } G^{(p)}$$

3. Obtain $G^{(d)} = (V, E^{(d)})$ by orienting as many edges in $G^{(p)}$ as possible without
   - creating immoralities
   - creating directed cycles
ad (1):

start with conditional independence graph \( G = (V, E) \)
\[ a \leftarrow b \iff X_a \perp X_b \mid X_{\{a,b\}} \]

search for sets \( S_{ab} \):

start with sets of cardinality 0, then cardinality 1, etc

remove edges as soon as separation is found

ad (3):

the following rules are required to obtain a maximally oriented graph:

\( R_1: \ a \rightarrow b \rightarrow c \Rightarrow a \rightarrow b \rightarrow c \)

\( R_2: \ a \xrightarrow{c} b \Rightarrow a \xrightarrow{c} b \)

\( R_3: \)

\( R_4: \)
Example

\[ \mathcal{J}(G) : \begin{align*}
    &1 \parallel 3 \parallel 2, 1 \parallel 4, 1 \parallel 8, 6, 7 \parallel 3, 2 \\
    &2 \parallel 4, 2 \parallel 8, 6, 7 \parallel 3 \\
    &3 \parallel 8, 7 \parallel 3, 14, 5 \parallel 3 \\
    &4 \parallel 7 \parallel 12 \\
    &5 \parallel 7 \parallel 6
\end{align*} \]

\[ \mathcal{G}^{(1)} : \begin{align*}
    &\begin{array}{c}
        1 \quad 2 \\
        \downarrow \quad \quad \downarrow \\
        3 \quad 4
    \end{array}
\end{align*} \]

\[ \mathcal{G}^{(2)} : \begin{align*}
    &\begin{array}{c}
        1 \quad 2 \\
        \downarrow \quad \quad \downarrow \\
        3 \quad 4
    \end{array}
\end{align*} \]

\[ \mathcal{G}^{(3)} : \begin{align*}
    &\begin{array}{c}
        1 \quad 2 \\
        \downarrow \quad \quad \downarrow \\
        3 \quad 4
    \end{array}
\end{align*} \]
Structures with bidirected edges

Aim: Find minimal consistent graphs in
\[ \mathcal{G} = \{ G \text{ directed graph without directed cycles} \} \]

For J(P) search for graph with 4 types of edges:

\[
\begin{align*}
& a \rightarrow b \quad \Rightarrow \quad a \rightarrow b \in E \\
& a \rightarrow b \quad \Rightarrow \quad a \rightarrow b \in E \lor a \leftarrow b \in E \\
& a \leftarrow b \quad \Rightarrow \quad a \leftarrow b \in E \\
& a \leftarrow b \quad \Rightarrow \quad a \leftarrow b \in E \lor a \leftarrow b \in E
\end{align*}
\]

\[ \text{etc.} \]
IC* algorithm (Inductive causation with latent variables)

(1) Find undirected graph $G^{u} = (V_{1}, E_{1})$

\[ a \rightarrow b \notin E_{1} \iff \exists S_{ab} \subseteq V_{1} \backslash \{a, b\}: X_{a} \perp\!\!\!\!\!\!\perp X_{b} \mid X_{S_{ab}} \mid P \]

(2) Search for immoralities $\rightarrow G^{w} = (V_{1}, E^{w})$

\[ a \rightarrow c \leftarrow b \text{ in } G^{u}, c \notin S_{ab} \Rightarrow a \rightarrow c \leftarrow b \text{ in } G^{w} \]

(3) Obtain $G^{*} = (V_{1}, E^{*})$ by orienting and marking as many edges as possible according to

\[ R_{1}^{*} : \begin{cases} a \rightarrow c \rightarrow b \\ a \rightarrow c \leftarrow b \end{cases} \Rightarrow a \rightarrow c^{*} \rightarrow b \]

\[ R_{2}^{*} : \begin{cases} a \rightarrow b \\ * \rightarrow c_{1} \rightarrow \cdots \rightarrow c_{n} \end{cases} \Rightarrow \begin{cases} a \rightarrow b \\ * \rightarrow c_{1} \rightarrow \cdots \rightarrow c_{n} \end{cases} \]

\[ \Rightarrow \text{ only simple graphs (no multiple edges)} \]
Example

\[ G_x: \]

\[ G^{(1)} \]

\[ G^{(2)} \]

\[ G^{(3)} \]

\[ G_x \]

\[ G^{(1)} \]

\[ G^{(2)} \]

\[ G^{(3)} \]