STAT391, Lecture 8

Market Models - The Lognormal Forward LIBOR model (LFM)

Introduction

We have seen that if \( \tilde{P} \) is a risk neutral measure, the bond price will under some suitable conditions follow

\[
dP(t,T) = r_t P(t,T) dt + P(t,T) v(t,T) d\tilde{W}(t), \quad t \leq T.
\]

Here and throughout, the Brownian motions are \( d \)-dimensional. If we define

\[
G(t,T,T^*) = \frac{P(t,T)}{P(t,T^*)}, \quad t \leq T \leq T^*, \tag{0.1}
\]

then

\[
G(t,T,T^*) = \frac{1}{F_{P(t,T,T^*)}(t,T)},
\]

where \( F_{P(t,T,T^*)}(t,T) \) is the forward price at time \( t \) of a \( T^* \)-bond delivered at time \( T \).

By the product formula,

\[
dG(t,T,T^*) = \frac{1}{P(t,T^*)}dP(t,T) + P(t,T) d\left( \frac{1}{P(t,T^*)} \right) + dP(t,T) d\left( \frac{1}{P(t,T^*)} \right). \tag{0.2}
\]

Now by Itô's formula,

\[
d \left( \frac{1}{P(t,T^*)} \right) = -\frac{1}{P^2(t,T^*)} dP(t,T^*) + \frac{1}{P^3(t,T^*)} (dP(t,T^*))^2
\]

\[
= \frac{1}{P(t,T^*)} \left( (-r_t + |v(t,T^*)|^2) dt - v'(t,T^*) d\tilde{W}(t) \right). \tag{0.3}
\]

Inserting this into (0.2) yields,

\[
dG(t,T,T^*) = G(t,T,T^*) \left( (|v(t,T^*)|^2 - v'(t,T) v(t,T^*)) dt
\]

\[
+ (v(t,T) - v(t,T^*))' d\tilde{W}(t) \right)
\]

\[
= G(t,T,T^*) \left( v(t,T) - v(t,T^*) \right)' \left( d\tilde{W}(t) - v(t,T^*) \right)
\]

\[
= G(t,T,T^*) \gamma'(t,T,T^*) dW^{T^*}(t), \tag{0.4}
\]

where

\[
\gamma(t,T,T^*) = v(t,T) - v(t,T^*), \tag{0.5}
\]

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and
\[ W^{T^*}(t) = \bar{W}(t) - \int_0^t v(s, T^*) ds \] (0.6)
is a Brownian motion under the measure \( P^{T^*} \) given by
\[ \frac{dP^{T^*}}{dP} \bigg|_{\mathcal{F}_t} = e^{-\frac{1}{2} \int_0^t \left( \int_0^s v(u, T^*) d\bar{W}(u) - \frac{1}{2} \int_0^s |v(u, T^*)|^2 du \right) ds} = \eta_t \]
say. Indeed, as in formula (0.6), Lecture 4, we have that
\[ \eta_t = \frac{P(t, T^*)}{B_t P(0, T^*)}. \] (0.7)
Actually, the measure \( P^{T^*} \) was introduced in Lecture 4 under the name forward measure. We saw in Exercise 2, HW 4, that the simply compounded (LIBOR) forward rate \( F(t, T, T^*) \) (see (0.14) below for the definition) is a martingale for \( t \leq T \) and any \( T < T^* \). Therefore, by Itô’s representation theorem there exists an adapted process \( \sigma(t, T, T^*) \) so that
\[ dF(t, T, T^*) = F(t, T, T^*) \sigma'(t, T, T^*) dW^{T^*}(t), \quad t < T < T^*. \] (0.8)
The forward model

We shall now take a different approach and just assume that there exists a measure \( P^{T^*} \) so that
\[ dG(t, T, T^*) = G(t, T, T^*) \gamma'(t, T, T^*) dW^{T^*}(t), \quad t < T < T^*, \]
where \( W^{T^*} \) is a \( d \)-dimensional Brownian motion under \( P^{T^*} \), \( \gamma(t, T, T^*) \) is an adapted process and \( G(t, T, T^*) \) is given by (0.1). Note that this assumption is of the form (0.4), the difference is that we do not give any bond price dynamics. Neither do we assume the existence of a risk neutral measure nor the existence of the risk free interest rate \( r_t \).

Next we define for \( \max\{U, T\} \leq T^* \),
\[ G(t, T, U) = \frac{G(t, T, T^*)}{G(t, U, T^*)}. \] (0.9)
Note that
\[ G(t, T, U) = \frac{P(t, T)}{P(t, U)}. \] (0.10)
Taking the differential in (0.9) and arguing as in (0.2)-(0.4) gives
\[ dG(t, T, U) = G(t, T, U)((\gamma(t, U, T^*) - \gamma'(t, T, T^*) \gamma(t, U, T^*)) dt + (\gamma(t, T, T^*) - \gamma(t, U, T^*))' dW^{T^*}(t)) = G(t, T, U) \gamma'(t, T, U) dW^{T^*}(t) - \gamma(t, U, T^*) dt) = G(t, T, U) \gamma'(t, T, U) dW^U(t), \] (0.11)
where
\[ \gamma(t, T, U) = \gamma(t, T, T^*) - \gamma(t, U, T^*), \]
and
\[ W^U(t) = W^{T^*}(t) - \int_0^t \gamma(s, U, T^*) ds \]
is a Brownian motion under the measure \( P^U \) given by
\[ \frac{dP^U}{dP^T} |_{\mathcal{F}_t} = e^{\int_0^t \gamma'(s, U, T^*) dW^{T^*}(s) - \frac{1}{2} \int_0^t \gamma^2(s, U, T^*) ds}. \]

Let \( t < T < U \leq T^* \) and let \( \tau = \tau(T, U) \), the time between \( T \) and \( U \). By Lecture 1, the forward LIBOR rate satisfies
\[ F(t, T, U) = \frac{1}{\tau} \left( \frac{P(t, T)}{P(t, U)} - 1 \right) = \frac{1}{\tau} (G(t, T, U) - 1), \]
so that
\[ G(t, T, U) = 1 + \tau F(t, T, U). \]

From (0.11) and (0.14),
\[
\begin{align*}
\frac{dF(t, T, U)}{\tau} &= \frac{1}{\tau} dG(t, T, U) \\
&= \frac{1}{\tau} G(t, T, U) \gamma'(t, T, U) dW^U(t) \\
&= \frac{1}{\tau} (1 + \tau F(t, T, U)) \gamma'(t, T, U) dW^U(t).
\end{align*}
\]

Now define \( \sigma(t, T, T^*) \) by
\[ \gamma(t, T, U) = \frac{\tau F(t, T, U)}{1 + \tau F(t, T, U)} \sigma(t, T, U). \]

Then
\[ dF(t, T, U) = F(t, T, U) \sigma'(t, T, U) dW^U(t), \quad t \leq T. \]

Note that (0.16) is the same as (0.8) (with \( U = T^* \)). When \( \gamma(t, T, U) \) is deterministic, as is the usual assumption in earlier lectures, (0.16) is not in lognormal form. However, if \( \sigma(t, T, U) \) is deterministic, then it is in lognormal form.

Now consider the discrete tenors \( T_0 < T_1 < \cdots < T_n = T^* \), and set
\[ \gamma(t, T_{i-1}, T_i) = \frac{\tau_i F(t, T_{i-1}, T_i)}{1 + \tau_i F(t, T_{i-1}, T_i)} \sigma(t, T_{i-1}, T_i), \]
where \( \tau_i = \tau(T_{i-1}, T_i) \). Then as in (0.16),
\[ dF(t, T_{i-1}, T_i) = F(t, T_{i-1}, T_i) \sigma'(t, T_{i-1}, T_i) dW^{T_i}(t), \quad t \leq T_{i-1}. \]
Note that we need different Brownian motions for different tenors (maturities). To see this, assume e.g. we could write
\[ dF(t, T_{i-1}, T_i) = F(t, T_{i-1}, T_i) \sigma(t, T_{i-1}, T_i) dW^T(t). \]

Then reversing the above steps, this would give
\[ dG(t, T_{i-1}, T_i) = G(t, T_{i-1}, T_i) \gamma(t, T_{i-1}, T_i) dW^T(t). \]

However, this is not consistent with (0.11) for \( T = T_{i-1} \) and \( U = T_i \), so the split up in different Brownian motions is necessary. This is in agreement with the results of Lecture 4, since the forward measures \( P^T \) and corresponding Brownian motions \( W^T \) introduced there were made to match the delivery date \( T \) of the contingent claims. The contingent claims to be priced here are the caplets

\[ X_i = \tau_i(L(T_{i-1}, T_i) - R_i)^+, \quad i = 1, \ldots, n, \]

where \( X_i \) has delivery time \( T_i \). Since by (0.1), \( F(t, T_{i-1}, T_i) \) is a martingale under \( P^{T_i} \), it is clear from e.g. Exercise 2, HW 4, that \( P^{T_i} \) is exactly the \( T_i \)-forward measure. Now \( L(T_{i-1}, T_i) = F(T_{i-1}, T_{i-1}, T_i) \), so if we assume that \( \sigma(t, T_{i-1}, T_i) \) is deterministic, a slight extension of Exercise 3c, HW 6, gives

\[ \Pi X_i(t, T_i) = (P(t, T_{i-1}) - P(t, T_i)) N(d_{1,i}) - \tau_i R_i P(t, T_i) N(d_{2,i}), \quad t < T_{i-1}, \quad (0.19) \]

where
\[ d_{1,i} = \log \left( \frac{F(t, T_{i-1}, T_i)}{R_i} \right) + \frac{1}{2} \sigma^2(t, T_{i-1}, T_i), \]
\[ d_{2,i} = d_{1,i} - b(t, T_{i-1}, T_i), \]
\[ b^2(t, T_{i-1}, T_i) = \int_t^{T_{i-1}} \sigma(s, T_{i-1}, T_i) ds. \]

Equation (0.18) gives the dynamics of the forward LIBOR rate \( F(t, T_{i-1}, T_i) \) under the measure \( P^{T_i} \). To find its dynamics under \( P^{T_k} \), we have from (0.13)
\[ dW^{T_i}(t) = dW^{T_n}(t) - \gamma(t, T_i, T_n) dt, \]
\[ dW^{T_k}(t) = dW^{T_n}(t) - \gamma(t, T_k, T_n) dt, \]

and subtraction of the second term from the first gives by (0.12),
\[ dW^{T_i}(t) = dW^{T_k}(t) - \gamma(t, T_i, T_k). \quad (0.20) \]

Assume that \( i < k \). Then using so-called telescoping of sums,
\[ \sum_{j=i+1}^{k} \gamma(t, T_{j-1}, T_j) = \sum_{j=i+1}^{k} (\gamma(t, T_{j-1}, T_n) - \gamma(t, T_j, T_n)) \]
\[ = \gamma(t, T_i, T_n) - \gamma(t, T_k, T_n) = \gamma(t, T_i, T_k). \]
Therefore, by (0.17) and (0.20),
\[ dW^T_i(t) = dW^T_k(t) - \sum_{j=i+1}^{k} \frac{\tau_j F(t, T_{j-1}, T_j)}{1 + \tau_j F(t, T_{j-1}, T_j)} \sigma(t, T_{j-1}, T_j) dt, \quad t \leq T_i. \]

If \( k < i \), it follows from (0.12) that
\[ \gamma(t, T_i, T_k) = -\gamma(t, T_k, T_i) = -\sum_{j=k+1}^{i} \gamma(t, T_{j-1}, T_j), \]
and thus
\[ dW^T_i(t) = dW^T_k(t) + \sum_{j=k+1}^{i} \frac{\tau_j F(t, T_{j-1}, T_j)}{1 + \tau_j F(t, T_{j-1}, T_j)} \sigma(t, T_{j-1}, T_j) dt, \quad t \leq T_k. \]

Inserting these expressions for \( dW^T_i(t) \) into (0.18) gives the dynamics of \( F(t, T_{i-1}, T_i) \) in terms of the \( P^T_k \) Brownian motions \( W^T_k \).

**The money account**

Since we are only looking at discrete times (discrete tenor) \( T_0, \ldots, T_n \), it is by no means clear how to define a money account \( B_t \) that is well defined for all \( t \). However, it can be defined on \( 0, T_0, \ldots, T_n \), and we shall define a risk neutral measure \( \hat{P} \) so that for \( i \leq k \leq n \),
\[ P(T_i, T_k) = \mathbb{E}[B_{T_k}^{-1}B_{T_i} | \mathcal{F}_{T_i}], \quad (0.21) \]

When extended to all \( t \), our method does not give a money account with finite variation, and therefore the notation \( \hat{P} \) is used instead of the standard \( \hat{P} \) for risk neutral measures.

To construct the money account, let \( B_0 = 1 \) and then recursively
\[ B_{T_j} = B_{T_{j-1}} G(T_{j-1}, T_{j-1}, T_j) = \frac{B_{T_{j-1}}}{P(T_{j-1}, T_j)}, \quad (0.22) \]

where \( T_{-1} = 0 \). This recursion is easily seen to give
\[ B_{T_j} = \frac{1}{\prod_{i=0}^{j} P(T_{i-1}, T_i)}. \quad (0.23) \]

It is clear from this that \( B_{T_j} \) is \( \mathcal{F}_{T_{j-1}} \) measurable, i.e. in the stochastic calculus vocabulary it is predictable. It is therefore well suited to serve as a money account on the discrete time points. The economic meaning of \( B \) is that we start at time 0 with one unit of money buying \( T_1 \)-bonds, and then at each time instant \( T_j \) the money is rolled over to \( T_{j+1} \) by buying \( T_{j+1} \)-bonds.
Now let \( \hat{P} \) be defined by

\[
\frac{d\hat{P}}{dP_{T_n}} \bigg|_{\mathcal{F}_{T_n}} = \frac{B_{T_j} P(0, T_n)}{P(T_j, T_n)}.
\] (0.24)

Note that (0.24) is in correspondence with (0.7). In particular we have

\[
\frac{d\hat{P}}{dP_{T_n}} \bigg|_{\mathcal{F}_{T_n}} = B_{T_n} P(0, T_n).
\]

We will now show that (0.24) really defines a probability measure on \((\Omega, \mathcal{F}_{T_n})\). It could have been obtained by (0.7) and then arguing backwards, but we shall give a direct proof. To do so, note that if \( \frac{dp^*}{dp} = \eta \), then

\[
P^*(A) = \int_A dP^* = \int_A \eta dP = \int_\Omega 1_A \eta dP = E[1_A \eta],
\]

and in particular \( \hat{P}(\Omega) = E[\eta] \). Another property we shall use is that if \( X \) and \( Y \) are stochastic variables with \( X \) \( \mathcal{G} \)-measurable, then

\[
E[XY] = E[E[XY|\mathcal{G}]] = E[XE[Y|\mathcal{G}]].
\]

Similarly if \( \mathcal{G} \subset \mathcal{F} \), then the tower property of conditional expectation gives

\[
E[XY|\mathcal{F}] = E[E[XY|\mathcal{G}]|\mathcal{F}] = E[XE[Y|\mathcal{G}]|\mathcal{F}].
\]

Using these we have

\[
\hat{P}(\Omega) = E^{T_n} \left[ \frac{B_{T_j} P(0, T_n)}{P(T_j, T_n)} \right]
\]

\[
= E^{T_n}[B_{T_j} P(0, T_n) G(T_j, T_j, T_n)]
\]

\[
= E^{T_n}[B_{T_j} P(0, T_n) E^{T_n} [G(T_j, T_j, T_n)|\mathcal{F}_{T_j-1}]]
\]

\[
= E^{T_n}[B_{T_j} P(0, T_n) G(T_{j-1}, T_j, T_n)]
\]

\[
= E^{T_n} \left[ B_{T_j} P(0, T_n) \frac{P(T_{j-1}, T_j)}{P(T_{j-1}, T_n)} \right]
\]

\[
= E^{T_n} \left[ B_{T_{j-1}} P(0, T_n) \right] \frac{P(T_{j-1}, T_j)}{P(T_{j-1}, T_n)}.
\]

Continuing like this, we finally arrive at

\[
\hat{P}(\Omega) = E^{T_n} \left[ \frac{B_0 P(0, T_n)}{P(0, T_n)} \right] = 1,
\]

and we are done. We shall prove in Exercise 1, HW 8, that (0.21) holds.
To extend this definition to continuous time, let

\[ a(t) = \min\{k : T_k \geq t\}. \]

This means that \( a(t) = k \) when \( T_{k-1} < t \leq T_k \). Therefore, \( T_{a(t)-1} < t \leq T_{a(t)} \). Also \( a(T_k) = k \) and so \( T_{a(T_k)} = T_k \). We now define the money account, or rather savings account by

\[ G_t = P(t, T_{a(t)})B_{a(t)}, \quad G_0 = 1. \]  

Clearly \( G \) is adapted. The economic meaning of \( G \) is again that of rolling money over, or to be more specific:

1. At time 0, buy \( \frac{1}{P(0,T_0)} = B_{T_0} \) of the \( T_0 \)-bonds and hold them to maturity \( T_0 \). Value at time \( T_0 \) is then \( B_{T_0} \).

2. At time \( T_0 \), buy \( \frac{B_{T_0}}{P(T_0,T_1)} = B_{T_1} \) of the \( T_1 \)-bonds and hold them to maturity \( T_1 \). Value at time \( T_1 \) is then \( B_{T_1} \).

3. Keep doing this, so at time \( T_j \), buy \( \frac{B_{T_j}}{P(T_j,T_{j+1})} = B_{T_{j+1}} \) of the \( T_{j+1} \)-bonds and hold them to maturity \( T_{j+1} \). Value at time \( T_{j+1} \) is then \( B_{T_{j+1}} \).

From this, if \( P(t, T_j) \) is continuous in \( t \), it is clear that \( G \) is continuous in \( t \) as well. Also for \( t \neq a(t) \),

\[ dG_t = B_{a(t)}dP(t, T_{a(t)}). \]  

However, \( G \) will typically not have finite variation, and it is therefore the name savings account is more apt than money account since the latter has finite variation in the examples we have seen so far. Also we study the cap market where \( \tau(T_{i-1}, T_i) \) is 3 or 6 months only, so \( G \) will represent a savings account well.

We shall now look more into the risk neutral measure \( \hat{P} \), and it will follow easily from our investigations that for \( t \leq T_j \),

\[ P(t, T_j) = \hat{E}[G_t G_{T_j}^{-1} | \mathcal{F}_t], \quad j = 0, 1, \ldots, n. \]  

Our main objective is to explore the relationship between \( \hat{P} \) and the forward measures \( P^T_i \). To do so, assume that there exists an objective probability measure \( P \) so that the bond dynamics follows

\[ dP(t, T_j) = m(t, T_j)P(t, T_j)dt + P(t, T_j)v(t, T_j)dW(t), \quad t \leq T_j \]  

for \( j = 0, \ldots, n \). Here \( m \) and \( v \) are adapted processes, and \( W \) is a Brownian motion with respect to \( P \).

Arguing as in (0.2)-(0.4), with

\[ Z_j(t) = \frac{P(t, T_j)}{G_t}, \quad j = 0, 1, \ldots, n, \]
we get from (0.26) and (0.28)

\[
d\!Z_j(t) = \frac{1}{G_t} dP(t, T_j) - P(t, T_j) \frac{1}{G_t^2} dP(t, T_{a(t)}) + P(t, T_j) \frac{1}{G_t^2} (dP(t, T_{a(t)}))^2 + dP(t, T_j) d \left( \frac{1}{G_t} \right) \\
= Z_j(t) (m(t, T_j) - m(t, T_{a(t)})) + v'(t, T_{a(t)}) (v(t, T_j) - v(t, T_{a(t)})) dt \\
+ Z_j(t) (v(t, T_j) - v(t, T_{a(t)})) dW(t).
\]

(0.29)

Since \( G_t \) is our money account, \( Z_j \) must be a (local) martingale under the risk neutral measure \( \hat{P} \) for all \( j = 0, 1, \ldots, n \). Now volatility is unchanged under an equivalent change of measure, hence (0.29) gives

\[
d\!Z_j(t) = Z_j(t) (v(t, T_j) - v(t, T_{a(t)})) d\hat{W}(t), \quad j = 0, 1, \ldots, n,
\]

where \( \hat{W} \) is a Brownian motion under \( \hat{P} \). Since \( \hat{P} \) is equivalent to \( P \), there must exist an adapted process \( \lambda_t \) so that

\[
\hat{W}(t) = W(t) - \int_0^t \lambda_s ds.
\]

(0.31)

Therefore, by Girsanov’s theorem, the measure \( \hat{P} \) is given by

\[
\frac{d\hat{P}}{dP} \bigg|_{\mathcal{F}_t} = e^{\int_0^t \lambda_s dW(s) - \frac{1}{2} \int_0^t \lambda_s^2 ds}.
\]

(0.32)

Comparing (0.29) and (0.31) we see that \( \lambda_t \) must satisfy

\[
m(t, T_j) - m(t, T_{a(t)}) - v'(t, T_{a(t)}) (v(t, T_j) - v(t, T_{a(t)})) = -\lambda_t' (v(t, T_j) - v(t, T_{a(t)})),
\]

which can be written as

\[
m(t, T_j) - m(t, T_{a(t)}) = (v(t, T_{a(t)}) - \lambda_t)' (v(t, T_j) - v(t, T_{a(t)}))
\]

for \( j = 0, 1, \ldots, n \). This gives a necessary relation between the bond price dynamics of bonds with different maturities. Note that for \( t < T_{j-1} \) it yields,

\[
m(t, T_i) - m(t, T_{i-1}) = m(t, T_i) - m(t, T_{a(t)}) - (m(t, T_{i-1}) - m(t, T_{a(t)})) \\
= (v(t, T_{a(t)}) - \lambda_t)' (v(t, T_i) - v(t, T_{i-1})).
\]

(0.33)

Formula (0.32) gives the relation between the measures \( \hat{P} \) and \( P \). However, since the modelling here is done with the forward measures \( P^{T_i} \), \( i = 0, 1, \ldots, n \), we are more interested in the relation between \( \hat{P} \) and \( P^{T_i} \). To find this relation, we start by using Itô’s formula on

\[
1 + \tau_i F(t, T_{i-1}, T_i) = \frac{P(t, T_{i-1})}{P(t, T_i)} = G(t, T_{i-1}, T_i)
\]

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to get, using the standard methods,
\[ dF(t, T_{i-1}, T_i) = F(t, T_{i-1}, T_i)\mu(t, T_{i-1}, T_i)dt + F(t, T_{i-1}, T_i)\sigma'(t, T_{i-1}, T_i)dW(t), \tag{0.34} \]

where
\[
\begin{align*}
\mu(t, T_{i-1}, T_i) &= H(t, T_{i-1}, T_i)(m(t, T_i) - m(t, T_{i-1}) - v'(t, T_i)(v(t, T_{i-1}) - v(t, T_i))), \\
\sigma(t, T_{i-1}, T_i) &= H(t, T_{i-1}, T_i)(v(t, T_{i-1}) - v(t, T_i)), \\
H(t, T_{i-1}, T_i) &= \frac{1}{\tau_i}G(t, T_{i-1}, T_i) = \frac{1 + \gamma_i F(t, T_{i-1}, T_i)}{\tau_i F(t, T_{i-1}, T_i)}.
\end{align*}
\]

Using (0.33) and the above definition for \( \sigma(t, T_{i-1}, T_i) \) gives,
\[
\begin{align*}
\mu(t, T_{i-1}, T_i) &= H(t, T_{i-1}, T_i)(v(t, T_{a(t)}) - v(t, T_i) - \lambda_i \gamma (v(t, T_{i-1}) - v(t, T_i))) \\
&= (v(t, T_{a(t)}) - v(t, T_i) - \lambda_i \gamma \sigma(t, T_{i-1}, T_i)). \tag{0.35}
\end{align*}
\]

Inserting this into (0.34) yields together with (0.31),
\[
\begin{align*}
dF(t, T_{i-1}, T_i) &= F(t, T_{i-1}, T_i)\sigma'(t, T_{i-1}, T_i)(v(t, T_{a(t)}) - v(t, T_i) - \lambda_i)dt + dW(t) \\
&= F(t, T_{i-1}, T_i)\sigma'(t, T_{i-1}, T_i)((v(t, T_{a(t)}) - v(t, T_i))dt + d\hat{W}(t)).
\end{align*}
\]

Comparing this with (0.18) gives (remember volatility remains unchanged),
\[
W(t) = \hat{W}(t) + \int_0^t (v(s, T_{a(s)}) - v(s, T_i))ds, \quad t \leq T_i. \tag{0.36}
\]

Therefore,
\[
\frac{d\hat{P}}{\hat{P}T_i} \bigg|_{\mathcal{F}_i} = e^{\int_0^t (v(s, T_{a(s)}) - v(s, T_i))dW(T_i) - \frac{1}{2} \int_0^t (v(s, T_{a(s)}) - v(s, T_i))^2ds ds}.
\]

It is interesting to compare (0.36) with (0.6), i.e. the Brownian motions under the risk neutral and the forward measure for the two different definitions of money accounts. We see that the difference is that (0.36) contains the extra term \( v(s, T_{a(s)}) \), which of course comes from the fact that the money account \( G_i \) is not of finite variation, as opposed to \( B_i \).

From the fact that \( \sigma(t, T_{i-1}, T_i) = H(t, T_{i-1}, T_i)(v(t, T_{i-1}) - v(t, T_i)) \), we get
\[
v(t, T_{a(t)}) - v(t, T_i) = \sum_{j=a(t)+1}^i (v(t, T_{j-1}) - v(t, T_j)) = \sum_{j=a(t)+1}^i \frac{1}{H(t, T_{j-1}, T_j)} \sigma(t, T_{j-1}, T_j).
\]

Using this together with the above definition of \( H(t, T_j, T_j) \) finally gives for \( t \leq T_{i-1} \),
\[
\begin{align*}
dF(t, T_{i-1}, T_i) &= F(t, T_{i-1}, T_i) \left( \sum_{j=a(t)+1}^i \frac{\tau_j F(t, T_{j-1}, T_j)}{1 + \tau_j F(t, T_{j-1}, T_j)} \sigma'(t, T_{j-1}, T_j) \right) \sigma(t, T_{i-1}, T_i)dt \\
&+ F(t, T_{i-1}, T_i)\sigma'(t, T_{i-1}, T_i)dW(t).
\end{align*}
\]
The proof of (0.27) is now quite easy. Let \( S_1(t) = P(t, T_j) \) and \( S_0(t) = G_t \). Then by Theorem 0.3, Lecture 6, setting \( Y = 1 \) gives

\[
\Pi_Y(t, T_j) = G_t \hat{E} \left[ \frac{1}{G_{T_j}} \big| \mathcal{F}_t \right] = \hat{E}[G_t G_{T_j}^{-1} | \mathcal{F}_t].
\]

On the other hand, it is clear that \( \Pi_Y(t, T_j) = P(t, T_j) \), and (0.27) is proved.

A natural generalization of (0.27) would be to have

\[
P(t, T) = \hat{E}[G_t G_T^{-1} | \mathcal{F}_t]
\]

for all \( t < T \). Note that this in a sense goes beyond our model, since we have only assumed the discrete tenor \( T_0, T_1, \ldots, T_n \). For (0.37) to be true, it is necessary that

\[
Z(t) = \frac{P(t, T)}{G_t}
\]

is a martingale under \( \hat{P} \). However, this is a necessary condition for the model to be consistent. Using the same arguments as those leading to (0.31), \( Z \) is a \( \hat{P} \) martingale if and only if

\[
m(t, T) - m(t, T_{a(t)}) = (v(t, T_{a(t)}) - \lambda_t)(v(t, T) - v(t, T_{a(t)}))
\]

for all \( t \), where

\[
dP(t, T) = m(t, T)P(t, T)dt + P(t, T)v'(t, T)dW(t), \quad t \leq T.
\]

If this holds, the proof of (0.37) is exactly like the above proof of (0.27).