STAT391, Lecture 6

Market portfolios and change of numeraire

We start with the same setup as in Lecture 5, i.e. we let \( X(t) = (X_1(t), \ldots, X_d(t))^\top \) be a \( d \)-dimensional stochastic process of observable, but not necessarily tradable quantities. Assume that the dynamics of \( X \) satisfies

\[
dX(t) = \mu(t, X(t))dt + \delta(t, X(t))dW(t),
\]

where \( W \) is an \( n \)-dimensional Brownian motion, \( \mu(t, x) \) a \( d \) vector and \( \delta(t, x) \) a \( d \times n \) matrix. It is assumed that \( \mu \) and \( \delta \) are such that (0.1) has a unique solution. We will let the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) be generated by \( W \). In addition we assume that the short rate of interest is of the form \( r_t = r(t, X(t)) \). As in Lecture 5, we may have \( r(t, x) = x_d \), in which case \( r \) follows an SDE.

To avoid extra notational complexities, we assume that we work under the risk neutral probability measure \( \bar{P} \), i.e. that \( P = \bar{P} \). For simplicity we just write \( P \) instead of \( \bar{P} \) and \( W \) instead of \( \bar{W} \). In case these measures are not equal, the process of going from \( P \) to \( \bar{P} \) is described in Theorem 0.1, Lecture 5. Furthermore, we shall assume that all conditions of Theorem 0.1, Lecture 5 are satisfied, and we will freely use results from that theorem.

In addition to the observable but not necessarily tradable process \( X \), we shall assume the existence of \( k + 1 \) tradable securities \( S(t) = (S_0(t), \ldots, S_k(t))^\top \). Since we are working under the risk neutral measure, it follows from Equation (0.25), Lecture 5, that the locally riskless return of \( S_i(t) \) equals \( r_i \) for all \( i = 0, 1, \ldots, k \). It is then assumed that \( S \) has the dynamics

\[
dS(t) = r_i S(t)dt + \text{diag}(S(t))\sigma(t, X(t))dW(t),
\]

where \( \text{diag}(S(t)) \) is a \( (k + 1) \times (k + 1) \) diagonal matrix with the elements of \( S \) on its diagonal. Written in terms of the elements, we get

\[
dS_i(t) = r_i S_i(t) + S_i(t)\sigma_i(t, X(t))dW(t).
\]

We may for example let \( S_k(t) = B(t) = e^{\int_0^t r_s ds} \), in which case \( \sigma_k(t, X(t)) = 0 \).

If \( Y = \Phi(X(T), S(T)) \) is an \( \mathcal{F}_T \)-measurable contingent claim, to be delivered at \( T \), we know from equation (0.20), Lecture 5, that the price of \( Y \) at time \( t \leq T \) equals

\[
\Pi_Y(t, T) = E \left[ \Phi(X(T), S(T))e^{-\int_t^T r(s, X(s)) ds} \bigg| \mathcal{F}_t \right].
\]

Consider now a numeraire process \( \beta \) given by

\[
d\beta(t) = \mu_\beta(t)dt + \sigma_\beta(t)dW(t),
\]

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and it is assumed that the process $\beta$ is always positive. Later we will let $\beta$ just equal the tradeable process $S_0(t)$, and then $\mu_\beta(t) = r_1 S_0(t)$ and $\sigma_\beta(t) = S_0(t) \sigma_0(t, X(t))$. However, for the time being, $\beta$ is just a nonnegative numeraire process. Using $\beta$ as a numeraire, the normalized economy $Z(t) = Z_0(t), \ldots, Z_k(t)$ is

$$Z_i(t) = \frac{S_i(t)}{\beta(t)}, \ i = 1, \ldots, k.$$ 

Here are some more definitions

**Definition 0.1** We have the following definitions.

- A portfolio strategy is any adapted, $k + 1$-dimensional process
  
  \[ h(t) = (h_0(t), \ldots, h_k(t)). \]

- The $S$-value process corresponding to $h$ is
  
  \[ V^S(t; h) = h'(t)S(t) = \sum_{i=0}^{k} h_i(t)S_i(t). \]

  It is $S$-self financing if

  \[ dV^S(t; h) = h'(t)dS(t) = \sum_{i=0}^{k} h_i(t)dS_i(t). \]

- The $Z$-value process corresponding to $h$ is

  \[ V^Z(t; h) = h'(t)Z(t) = \sum_{i=0}^{k} h_i(t)Z_i(t). \]

  It is $Z$-self financing if

  \[ dV^Z(t; h) = h'(t)dZ(t) = \sum_{i=0}^{k} h_i(t)dZ_i(t). \]

- An $\mathcal{F}_T$-measurable contingent claim is said to be $S$-attainable if there exists an $S$-self financing portfolio strategy $h$ so that

  \[ V^S(T; h) = Y. \]

  It is $Z$-attainable if there exists a $Z$-self financing portfolio strategy $h$ so that

  \[ V^Z(T; h) = Y. \]
We have the following easy, but important result

**Theorem 0.1** With the above notation

a) The value processes \( V^S \) and \( V^Z \) are connected by

\[
V^Z(t; h) = \frac{1}{\beta(t)} V^S(t; h).
\]

b) A portfolio is \( S \)-self financing if and only if it is \( Z \)-self financing.

c) An \( \mathcal{F}_T \)-measurable contingent claim is \( S \)-attainable if and only if the \( \mathcal{F}_T \)-measurable contingent claim

\[
\frac{Y}{\beta(T)}
\]

is \( Z \)-attainable.

**Proof** For part a, just note that

\[
V^Z(t; h) = h'(t)Z(t) = \frac{1}{\beta(t)} h'(t)S(t) = \frac{1}{\beta(t)} V^S(t; h).
\]

For part b, assume that the portfolio \( h \) is \( S \)-self financing. Then we have (dropping the function argument \( t \)),

\[
\begin{align*}
Z &= \beta^{-1}S \\
V^S &= h'S \\
V^Z &= \beta^{-1}V^S \\
dV^S &= h'dS.
\end{align*}
\]

We must prove

\[
dV^Z = h'dZ. \tag{0.4}
\]

The rule for differentiating a product gives

\[
dZ = d(\beta^{-1}S) = \beta^{-1}dS + Sd\beta^{-1} + dSd\beta^{-1}.
\]

Therefore (0.4) follows because

\[
\begin{align*}
h'dZ &= h'(\beta^{-1}dS + Sd\beta^{-1} + dSd\beta^{-1)} \\
&= \beta^{-1}dV^S + V^Sd\beta^{-1} + dV^Sd\beta^{-1} \\
&= d(\beta^{-1}V^S) = dV^Z.
\end{align*}
\]

A similar argument applies if we assume that the portfolio \( h \) is \( Z \)-self financing.
Finally for part c, if $Y$ is $S$-attainable, then there is an $S$-self financing portfolio $h$ so that $V^S(T; h) = Y$. However, by part b, this portfolio will be self financing for $V^Z(T; h)$, but from part a,

$$V^Z(T; h) = \frac{V^S(T; h)}{\beta(T)} = \frac{Y}{\beta(T)}.$$

The other way is proved in an identical way.

Since arbitrage portfolios and attainable contingent claims are defined in terms of self financing portfolios, the last theorem gives

**Theorem 0.2** With the above notation and definitions

a) The $S$-market is arbitrage free if and only if the $Z$-market is arbitrage free.

b) The $S$-market is complete if and only if the $Z$-market is complete.

c) A price process $\Pi(t)$ is an arbitrage free process in the $S$-market if and only if the the process $\frac{\Pi(t)}{\beta(t)}$ is an arbitrage free process in the $Z$-market.

d) The following relationship holds between price processes for a contingent claim $Y$

$$\Pi_Y(t, T) = \beta(t)\Pi_{Y/\beta(T)}(t, T).$$

**Proof** For part a, if starting with nothing in the $S$-market can give $Y$ at time $T$, where $Y \geq 0$ and $P(Y > 0) > 0$, then the same strategy gives $\frac{Y}{\beta(T)}$ in the $Z$-market, and since $\beta(T) > 0$, the result follows since this argument also works in the opposite direction. Parts b and c is a straightforward application of Theorem 0.1. For part d, by definition $\Pi_Y(t, T) = \Pi^Y_S(t, T)$ is an arbitrage free price process for $Y$ in the $S$-market, hence by part c, $\frac{\Pi_Y(t, T)}{\beta(t)}$ is an arbitrage free price process for $\frac{Y}{\beta(T)}$ in the $Z$-market. But $\Pi^Z_{Y/\beta(T)}(t, T)$ is also an arbitrage free price process for $\frac{Y}{\beta(T)}$ in the $Z$-market, hence these two processes must be equal, i.e.

$$\Pi_Y(t, T) = \beta(t)\Pi^Z_{Y/\beta(T)}(t, T).$$

From now on we let the numeraire process $\beta$ be equal to $S_0$, so in particular it is assumed that $S_0(t) > 0$ for all $t$. Note also that this choice of numeraire gives $Z_0(t) = 1$.

Assume that the process $\eta$ given by
d

$$d\eta_t = \eta_t\sigma_0(t, X(t))dW(t), \quad \eta_0 = 1$$

is a martingale on $[0, T]$. A sufficient condition is the usual Novikov condition

$$E\left[e^{\frac{1}{2}\int_0^T \eta^2_s ds}\right] < \infty.$$
The solution of the SDE for \( \eta \) is as always
\[
\eta_t = e^{\int_0^t \sigma_0(s, X(s)) \, dW(s) - \frac{1}{2} \int_0^t |\sigma_0(s, X(s))|^2 \, ds},
\]
and by defining the measure \( P^0 \) by
\[
\frac{dP^0}{dP} |_{\mathcal{F}_0} = \eta_t, \tag{0.5}
\]
we know from Girsanov’s theorem that \( W^0 \) given by
\[
W^0(t) = W(t) - \int_0^t \sigma_0'(s, X(s)) \, ds \tag{0.6}
\]
is a Brownian motion under \( P^0 \). Note that the definition of \( P^0 \) only depends on the numeraire \( S_0 \). It is independent of delivery time \( T \), and it is the same for all assets.

We can now state the important theorem.

**Theorem 0.3** Let the numeraire be the price process of a traded asset \( S_0 \) with \( S_0(t) > 0 \) for all \( t \). Let the measure \( P^0 \) be defined by (0.5), so that in particular the process \( W^0 \) given by (0.6) is a Brownian motion under \( P^0 \). Also let the process \( Z \) be given by
\[
Z_i(t) = \frac{S_i(t)}{S_0(t)}, \quad i = 0, \ldots, k.
\]

We then have.

a) Under \( P^0 \), the dynamics of the \( Z \)-processes is given by
\[
dZ_i(t) = Z_i(t)(\sigma_i(t, X(t)) - \sigma_0(t, X(t))) \, dW^0(t), \quad i = 0, \ldots, k.
\]

b) Assume that \( Z \) is a martingale. A sufficient condition for this to hold is again the Novikov condition
\[
E \left[ e^{\frac{1}{2} \int_0^T |\sigma_i(t, X(t)) - \sigma_0(t, X(t))|^2 \, ds} \right] < \infty, \quad i = 0, \ldots, k.
\]

Then for every attainable \( T \)-claim \( Y \),
\[
\Pi_Y(t, T) = S_0(t)E^0 \left[ \frac{Y}{S_0(T)} \bigg| \mathcal{F}_t \right],
\]
where \( E^0 \) denotes expectation under \( P^0 \).

c) Similarly the \( P^0 \) dynamics of \( S \) is given by
\[
dS_i(t) = S_i(t)(r_i + \sigma_i(t, X(t))\sigma_0'(t, X(t))) \, dt + S_i(t)\sigma_i(t, X(t)) \, dW^0(t), \quad i = 0, \ldots, k.
\]
d) Finally the $P^0$ dynamics of $X$ is given by

$$dX(t) = (\mu(t, X(t)) + \delta(t, X(t))\sigma'_0(t, X(t)))dt + \delta(t, X(t))dW^0(t).$$

**Proof** We will use the following identification. Let $A$ and $B$ be adapted process, $A$ one dimensional while $B$ $n$ dimensional. Consider the SDE

$$dV(t) = V(t)A(t)dt + V(t)B'(t)dW(t).$$

Then using Itô’s formula on $\log(V(t))$, we get that

$$V(t) = V(0)e^{\int_0^t (A(s) - \frac{1}{2}B(s)^2)ds + \int_0^t B'(s)dW(s)},$$

(0.7)

where of course $B'(t)$ means the transpose of $B(t)$.

Now to the proof of part a. For simplicity we shall write $\sigma_i(t)$ for $\sigma_i(t, X(t))$. Then by (0.2), (0.6) and (0.7)

$$S_i(t) = S_i(0)e^{\int_0^t (r_s - \frac{1}{2}|\sigma_i(s)|^2)ds + \int_0^t \sigma_i(s)dW(s)}$$

$$= S_i(0)e^{\int_0^t (r_s + \sigma_i(s)\sigma'_0(s) - \frac{1}{2}|\sigma_i(s)|^2)ds + \int_0^t \sigma_i(s)dW^0(s)}.$$

Therefore

$$Z_i(s) = S_i(0)e^{\int_0^t (r_s + \sigma_i(s)\sigma'_0(s) - \frac{1}{2}|\sigma_i(s)|^2)ds + \int_0^t \sigma_i(s)dW^0(s)}$$

$$\times (S_0(0))^{-1}e^{-\int_0^t (r_s + \frac{1}{2}|\sigma_0(s)|^2)ds - \int_0^t \sigma_0(s)dW^0(s)}$$

$$= Z_i(0)e^{-\frac{1}{2}\int_0^t |\sigma_i(s) - \sigma_0(s)|^2ds + \int_0^t (\sigma_i(s) - \sigma_0(s))dW^0(s)}.$$

Thus by (0.7),

$$dZ_i(t) = Z_i(t)(\sigma_i(t) - \sigma_0(t))dW^0(t),$$

and this shows part a. By what we have just proved, and the assumption that $Z$ is a martingale under $P^0$, not just a local martingale, gives by (0.3) and Theorem 0.2, part d,

$$\Pi_Y(t, T) = S_0(t)\Pi_{Y/S_0(t)}^Z(t, T) = S_0(t)E^0\left[\frac{Y}{S_0(T)}\bigg|\mathcal{F}_t\right].$$

The second equality follows since in the $Z$-market the process $Z$ is a martingale, hence the $r$ process in (0.3) is zero in the $Z$-market.

To prove part c, just combine the expressions for $S_0(t)$ and $Z_i(t)$ found above to get

$$S_i(t) = S_0(t)Z_i(t) = S_i(0)e^{\int_0^t (r_s + \sigma_i(s)\sigma'_0(s) - \frac{1}{2}|\sigma_i(s)|^2)ds + \int_0^t \sigma_i(s)dW^0(s)}.$$

Therefore, by (0.7)

$$dS_i(t) = S_i(t)(r_t + \sigma_i(t)\sigma'_0(t))dt + S_i(t)\sigma_i(t)dW^0(t),$$

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and part c is proved. Part d is easily obtained by inserting the definition of $W^0$ in (0.6) into (0.1). This ends the proof.

**Example 1** Consider the situation with $n = k + 1 = 2$, and assume we have two stocks $S_0(t)$ and $S_1(t)$. We are interested in pricing the European contingent claim of the form $Y = \Phi(S_0(T), S_1(T))$, where $\Phi$ is homogeneous of degree one, i.e. $\Phi(ax, ay) = a\Phi(x, y)$. Then with $Z_1(t) = \frac{S_1(t)}{S_0(t)}$, we get

$$Y = S_0(T)\Phi(1, Z_1(T)).$$

By Theorem 0.3, part b, the price of $Y$ at time $t$ equals

$$\Pi_Y(t, T) = S_0(t)E^0[\Phi(1, Z_1(T))|\mathcal{F}_t].$$

Now assume that under the risk neutral measure $P$, $S_1$ and $S_2$ follow (see Example 1, Lecture 5),

$$dS_i(t) = r_iS_i(t)dt + S_i(t)\sigma_i dW(t), \quad i = 0, 1,$$

where $\sigma_1$ and $\sigma_2$ are constant 2-vectors. Then by Theorem 0.3, part a,

$$Z_1(T) = Z_1(t)e^{-\frac{1}{2}[\sigma_1-\sigma_0]'^{\prime\prime}(T-t) + [\sigma_1-\sigma_0]'(W^0(t) - W^0(0))}. \quad (0.8)$$

Now given $\mathcal{F}_t$, $(\sigma_1 - \sigma_0)'(W^0(T) - W^0(t)) \sim \mathcal{N}(0, |\sigma_1 - \sigma_0|^2(T-t))$ under $P^0$, and therefore by Theorem 0.3, part a,

$$\Pi_Y(t, T) = S_0(t)\int_{-\infty}^{\infty} \Phi \left( 1, \frac{S_1(t)}{S_0(t)}e^{-\frac{1}{2}|\sigma_1-\sigma_0|^2(T-t) + |\sigma_1-\sigma_0|\sqrt{T-t}y} \right) \frac{1}{2\pi} e^{-\frac{1}{2}y^2} dy.$$

It is worth noting that this formula is correct even when the short rate $r_t$ is stochastic. We could for example have

$$dr_t = \mu(t, r_t)dt + \sigma'(t, r_t)d\hat{W}(t),$$

where $\hat{W}(t) = (W'(t), W_3(t))'$ is a 3-dimensional Brownian motion under $P$.

Now let us look at the exchange option discussed in Example 1, Lecture 5. To be in accordance with the above notation, we denote $S_2(t)$ by $S_0(t)$ and similarly $\sigma_2$ by $\sigma_0$. Then

$$Y = (S_1(T) - KS_0(T))^+ = S_0(T)(Z_1(T) - K)^+.$$

Therefore, by the above,

$$\Pi_Y(t, T) = S_0(t)E^0[(Z_1(T) - K)^+|\mathcal{F}_t] = S_0(t)E^0[Z_1(T)1_{[Z_1(T) > K]}|\mathcal{F}_t] - KS_0(t)P^0[Z_1(T) > K|\mathcal{F}_t].$$
It follows from (0.8) that $Z_1(T) = e^X$, where conditioned on $\mathcal{F}_t$, $X \sim \mathcal{N}(\mu, \sigma^2)$ under $P^0$. Here

$$
\mu = \log Z_1(t) - \frac{1}{2}\left| \sigma_1 - \sigma_0 \right|^2(T - t),
$$

$$
\sigma^2 = \left| \sigma_1 - \sigma_0 \right|^2(T - t).
$$

Using that

$$
S_0(t)e^{\mu + \frac{1}{2}\sigma^2} = S_0(t)Z_1(t)e^0 = S_1(t),
$$

Lemma 0.1, Lecture 4, gives that

$$
\Pi_Y(t, T) = S_1(t)N\left( \frac{\log \left( \frac{S_1(t)}{K_{S_0(t)}} \right) + \frac{1}{2}\left| \sigma_1 - \sigma_0 \right|^2(T - t)}{\left| \sigma_1 - \sigma_0 \right|\sqrt{T - t}} \right)
$$

$$
- S_0(t)N\left( \frac{\log \left( \frac{S_1(t)}{K_{S_0(t)}} \right) - \frac{1}{2}\left| \sigma_1 - \sigma_0 \right|^2(T - t)}{\left| \sigma_1 - \sigma_0 \right|\sqrt{T - t}} \right)
$$

**Example 2** Assume that under the risk neutral measure $P$ the dynamics of a $T$-bond follows

$$
dP(t, T) = r_t P(t, T) dt + P(t, T) v'(t, T) dW(t)
$$

and the dynamics of a stock follows

$$
dS_t = r_t S_t dt + S_t \sigma'(t) dW(t).
$$

Here $W$ is a 2-dimensional Brownian motion, and $v(t, T)$ and $\sigma(t)$ are two 2-dimensional vectors. We assume that both $v(t, T)$ and $\sigma(t)$ are nonstochastic.

Our aim is to price a European call option on the stock. Delivery time is $T$ and exercise price is $K$, so that

$$
Y = (S_T - K)^+.
$$

The price of $Y$ at time $t$ equals

$$
\Pi_Y(t, T) = E\left[ e^{-\int_t^T r_s ds} (S_T - K)^+ | \mathcal{F}_t \right]
$$

$$
= E\left[ e^{-\int_t^T r_s ds} S_T 1_{\{S_T > K\}} | \mathcal{F}_t \right] - KE\left[ e^{-\int_t^T r_s ds} 1_{\{S_T > K\}} | \mathcal{F}_t \right]
$$

$$
= \alpha_t - K \beta_t
$$

We will now calculate $\alpha_t$ and $\beta_t$ separately, using different changes of numeraire.

To find $\alpha_t$, let $S_0(t) = S_t$ and $S_1(t) = P(t, T)$, giving $Z_1(t) = \frac{P(t, T)}{S_t}$. Then Theorem 0.3a gives,

$$
\alpha_t = S_tE^0[1_{\{S_T > K\}} | \mathcal{F}_t]
$$
\begin{align*}
  &= S_t P^0(S_T > K \mid \mathcal{F}_t) \\
  &= S_t P^0 \left( \frac{1}{S_T} < \frac{1}{K} \mid \mathcal{F}_t \right) \\
  &= S_t P^0 \left( \frac{P(T, T)}{S_T} < \frac{1}{K} \mid \mathcal{F}_t \right) \\
  &= S_t P^0 \left( Z_1(T) < \frac{1}{K} \mid \mathcal{F}_t \right).
\end{align*}

By Theorem 0.3b,
\[ dZ_1(t) = Z_1(t)(v(t, T) - \sigma(t))'dW^0(t), \]
hence as in (0.8),
\[ Z_1(T) = Z_1(t) e^{-\frac{1}{2} \int_t^T |v(s, T) - \sigma(s)|^2 ds + \int_t^T (v(s, T) - \sigma(s))'dW^0(s)}. \tag{0.9} \]

Therefore, we can write \( Z_1(T) = e^{X} \), where \( X \) given \( \mathcal{F}_t \) follows \( X \sim \mathcal{N}(\mu, a^2(t, T)) \) under \( P^0 \), with
\begin{align*}
  \mu &= \log Z_1(t) - \frac{1}{2} a^2(t, T), \\
  a^2(t, T) &= \int_t^T |v(s, T) - \sigma(s)|^2 ds.
\end{align*}

This yields,
\[ \alpha_t = S_t P^0(X \leq -\log K \mid \mathcal{F}_t) = S_t N \left( \frac{\log \left( \frac{S_t}{KP(T, T)} \right) + \frac{1}{2} a^2(t, T)}{a(t, T)} \right). \]

To calculate \( \beta_t \), let \( S_0(t) = P(t, T) \) and \( \hat{S}_t(t) = S_t \). Since this is the opposite of what we did when working with \( \alpha_t \), the hat is added to observe the difference. We shall also write \( \hat{P}^0 \) as well as \( \hat{E}^0 \) to observe the difference of the two changes of numeraire employed here and above. Then since \( P(T, T) = 1 \),
\begin{align*}
  \beta_t &= P(t, T) \hat{E}^0[1_{\{S_T > K\}} \mid \mathcal{F}_t] \\
  &= P(t, T) \hat{P}^0(S_T > K \mid \mathcal{F}_t) \\
  &= P(t, T) \hat{P}^0 \left( \frac{S_T}{P(T, T)} > K \mid \mathcal{F}_t \right) \\
  &= P(t, T) \hat{P}^0(\hat{Z}_1(T) > K \mid \mathcal{F}_t).
\end{align*}

Again by Theorem 0.3b,
\[ d\hat{Z}_1(t) = \hat{Z}_1(t)(\sigma(t) - v(t, T))'d\hat{W}^0(t), \]
where \( \hat{W}^0 \) is a 2-dimensional Brownian motion w.r.t. the measure \( \hat{P}^0 \). Now as in (0.9), \( \hat{Z}_1(T) = e^{\hat{X}} \), where given \( \mathcal{F}_t \), \( \hat{X} \) has the \( \hat{P}^0 \) distribution \( \hat{X} \sim \mathcal{N}(\hat{\mu}, a^2(t, T)) \) with
\[
\hat{\mu} = \log \hat{Z}_1(t) - \frac{1}{2}a^2(t, T),
\]
and \( a^2(t, T) \) is as above. Then Lemma 0.1a, Lecture 4, gives
\[
\beta_t = P(t, T)\hat{P}^0(\hat{X} > \log K|\mathcal{F}_t) = P(t, T)N \left( \frac{\log \left( \frac{S_t}{KP(t, T)} \right) - \frac{1}{2}a^2(t, T)}{a(t, T)} \right).
\]
Note that the change of numeraire employed to calculate \( \beta_t \) is nothing more than going from the risk neutral measure to the forward measure \( P^F \) introduced in Lecture 4, so that \( \hat{P}^0 = P^F \). To summarize we have.

**Theorem 0.4** Assume that under the risk neutral measure \( P \) the dynamics of a \( T \)-bond follows
\[
dP(t, T) = r_tP(t, T)dt + P(t, T)v'(t, T)dW(t)
\]
and the dynamics of a stock follows
\[
dS_t = r_tS_tdt + S_t\sigma'(t)dW(t).
\]
Here \( W \) is a 2-dimensional Brownian motion, and \( v(t, T) \) and \( \sigma(t) \) are two 2-dimensional vectors. We assume that both \( v(t, T) \) and \( \sigma(t) \) are nonstochastic.

Then the price of a European call option on the stock, exercise price \( K \) and delivery time \( T \), i.e. on
\[
Y = (S_T - K)^+,
\]
equals
\[
\Pi_Y(t, T) = S_tN(d(t, T)) - KP(t, T)N(d(t, T) - a(t, T)),
\]
where
\[
d(t, T) = \frac{\log \left( \frac{S_t}{KP(t, T)} \right) + \frac{1}{2}a^2(t, T)}{a(t, T)},
\]
\[
a^2(t, T) = \int_t^T |v(s, T) - \sigma(s)|^2ds.
\]