STAT 391 HOMEWORK ASSIGNMENT 7

Exercise 1

For this exercise, it is sufficient to use the theory of forward measures from Lecture 4.

Let \( t < S < T_1 < T_2 \) and consider the \( \mathcal{F}_S \)-measurable contingent claim with value

\[
Y = (F_{P(T_1,T_2)}(S,T_1) - K)^+
\]

at delivery time \( S \). This means that if the forward price at time \( S \) for delivery at \( T_1 \) of a \( T_2 \)-bond is higher than \( K \), the holder of the contract gets the difference of that forward price and \( K \). For this to make any sense, it is clear that \( K < 1 \).

Assume the bond prices have the dynamics

\[
dP(t,T) = r_t P(t,T) dt + P(t,T) v(t,T) dW_t
\]

under the risk neutral measure \( P = \tilde{P} \). Here \( v(t,T) \) is nonstochastic for all \((t,T)\).

a) Show that the price of \( Y \) at time \( t < S \) equals

\[
\Pi_Y(t,T) = \frac{P(t,S)P(t,T_2)}{P(t,T_1)} e^{-c(t,S,T_1,T_2) N(d_1) - K P(t,S) N(d_1 - b(t,S,T_1,T_2))},
\]

where

\[
d_1 = \frac{\log \left( \frac{P(t,T_2)}{P(t,T_1)} \right) + \frac{1}{2} b^2(t,S,T_1,T_2) - c(t,S,T_1,T_2)}{b(t,S,T_1,T_2)},
\]

\[
b^2(t,S,T_1,T_2) = \int_t^S \gamma^2(s,T_1,T_2),
\]

\[
c(t,S,T_1,T_2) = \int_t^S \gamma(s,T_1,T_2) \gamma(s,S,T_1) ds,
\]

\[
\gamma(s,T_1,T_2) = v(s,T_2) - v(s,T_1).
\]

Hint: Use that \( \Pi_Y(t,T) = P(t,S)E^S[Y | \mathcal{F}_t] \) where \( P^S \) is the \( S \)-forward measure. Now we have from Lecture 4 that

\[
F_{P(T_1,T_2)}(S,T_1) = F_{P(T_1,T_2)}(t,T_1) e^{-\frac{1}{2} \int_t^S \gamma^2(s,T_1,T_2) ds + \int_t^S \gamma(s,T_1,T_2) dW_s^{T_1}}.
\]

This expression involves the \( P^{T_1} \) Brownian motion \( W^{T_1} \). However, \( W^{T_1} \) is not a \( P^S \) Brownian motion, so you must make the proper transfer from \( W^{T_1} \) to \( W^S \).

b) Now price the corresponding future contract

\[
Y = (f_{P(T_1,T_2)}(S,T_1) - K)^+.
\]
Assume instead that the bond price dynamics was
\[ dP(t, T) = r_t P(t, T)dt + P(t, T)v'(t, T)dW(t), \]
where \( v(t, T) \) is a nonstochastic \( n \)-vector for all \( (t, T) \) and \( W \) is an \( n \)-dimensional Brownian motion.

c) Explain how you can modify your answer in parts a and b above to accommodate this change in model, and discuss briefly the validity of this modification.

Exercise 2

We assume the same model as in Example 2, Lecture 6. So let the bond price dynamics under the risk neutral probability measure \( P \) be given by
\[ dP(t, T) = r_t P(t, T)dt + P(t, T)v'(t, T)dW(t) \]
and the dynamics of a stock follows
\[ dS_t = r_t S_t dt + S_t \sigma'(t)dW(t). \]
Here \( W \) is an \( n \)-dimensional Brownian motion, and \( v(t, T) \) and \( \sigma(t) \) are two \( n \)-dimensional vectors. We assume that both \( v(t, T) \) and \( \sigma(t) \) are nonstochastic.

We will price two different minimum return European stock options, both with time of delivery \( T \).

First consider the case where the minimum return equals the return on a bond, multiplied by a potential “safety” factor \( K \). The option value at time of delivery \( T \) thus equals
\[ Y = \left( S_T - K \frac{S_t}{P(t, T)} \right)^+. \]
If \( K = 1 \), the holder of the option and a stock is guaranteed a minimum return equal to a return on a bond.

a) Find \( \Pi(t, T) \), the value of \( Y \) at time \( t < T \).

b) Do the computations necessary in part a when \( \sigma(s) = (\sigma_1, \sigma_2)' \), a constant, and the forward rate dynamics is that of a mixed Ho-Lee and Hull-White model, i.e.
\[ df(t, T) = m(t, T)dt + \tau(t, T)dW(t), \]
with \( \tau(t, T) = (\tau_1, \tau_2 e^{-a(T-t)})' \). Note that here \( n = 2 \).

c) Now replace the bond in part a with the money account, so that
\[ Y = \left( S_T - KS_t e^{\int_t^T r_s ds} \right)^+. \]
Find \( \Pi_Y(t, T) \) in this case.
Exercise 3

Captions In this exercise we will discuss how we can price a European call option on a caplet. A caption is an option on a cap, but here we shall restrict ourselves to an option on a caplet, which is much easier. Assume the bond dynamics is given as

$$dP(t, T) = r_t P(t, T)dt + P(t, T)v'(t, T)dW(t),$$

where $W$ is an $n$-dimensional Brownian motion, and $v(t, T)$ is a two $n$-dimensional vectors. We assume that $v(t, T)$ is nonstochastic.

Let $t < S < T_1 < T_2$, and consider the $T_2$ deliverable caplet

$$X = \tau(L(T_1, T_2) - R)^+, \quad \tau = \tau(T_1, T_2),$$

where the time between $T_1$ and $T_2$. By Exercise 2, HW 5, the value of $X$ at time $S$ equals

$$\Pi_X(S, T_2) = P(S, T_1)N(\hat{d}) - (1 + \tau R)P(S, T_2)N(\hat{d} - b_1),$$

where

$$\hat{d} = \frac{\log \left( \frac{P(S, T_1)}{P(S, T_2)} \right) - \log(1 + \tau R) + \frac{1}{2} \beta^2_1}{\beta_1},$$

$$b_1^2 = \int_S^{T_1} |v(s, T_2) - v(s, T_1)|^2 ds.$$

Now consider our caplet which is a European call option on $\Pi_X(S, T_2)$, delivery time $S$ and exercise price $K$, i.e. on the $S$-deliverable

$$Y = (\Pi_X(S, T_2) - K)^+.$$

Show that the price of $Y$ at time $t < S$, $\Pi_Y(t, S)$, equals

$$\Pi_Y(t, S) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (g(z) - P(t, T_2)K)^+ e^{-\frac{1}{2}z^2} dz,$$

where

$$g(z) = P(t, T_1)e^{-\frac{1}{2}b_2^2 + b_2 z}N(h(z)) - (1 + \tau R)P(t, T_2)N(h(z) - b_1),$$

with

$$h(z) = \frac{\log \left( \frac{P(t, T_1)}{P(t, T_2)} \right) - \log(1 + \tau R) - \frac{1}{2} b_2^2 + \frac{1}{2} b_1^2 + b_1 z}{b_1},$$

$$b_2^2 = \int_t^S |v(s, T_2) - v(s, T_1)|^2 ds.$$