Exercise 1

Let \( t < S < T_1 < T_2 \) and consider a European exchange option which at delivery time \( S \) is worth
\[
Y = (P(S, T_1) - KP(S, T_2))^+, 
\]
with \( K > 1 \).

a) Define the constant \( R \) by \( K = 1 + \tau R \), where \( \tau = \tau(T_1, T_2) \) is the time between \( T_1 \) and \( T_2 \). Show that
\[
Y = \tau P(S, T_2)(F(S, T_1, T_2) - R)^+, 
\]
and use this to give an alternative verbal interpretation of the option.

b) Consider the opposite European exchange option which at delivery time \( S \) is worth
\[
Z = (KP(S, T_2) - P(S, T_1))^+. 
\]

Prove the Put-Call parity
\[
\Pi_Z(t, S) - \Pi_Y(t, S) = KP(t, T_2) - P(t, T_1). 
\]

Assume that under the risk neutral measure \( P = \tilde{P} \),
\[
dP(t, T) = r_t P(t, T)dt + P(t, T)v(t, T)dW(t), 
\]
where \( v(t, T) \) is a nonstochastic \( n \)-vector for all \( (t, T) \) and \( W \) is an \( n \)-dimensional Brownian motion.

c) Find the price \( \Pi_Z(t, S) \) of \( Z \) at time \( t \). Here it can be a good idea to write \( Z \) as
\[
Z = KP(S, T_1) \left( \frac{P(S, T_2)}{P(S, T_1)} - \frac{1}{K} \right)^+. 
\]
Exercise 2

In this exercise we will briefly discuss swaps, swaptions and caplets. Let $t \leq T_0 < T_1 < \cdots < T_n$, and let $\tau_i = \tau(T_{i-1} - T_i)$ be the time between $T_1$ and $T_{i-1}$. A PFS (Payer Forward Start) swap is a contract that at time $T_i$, $i = 1, \ldots, n$ pays $Y_i = \tau_i(L(T_{i-1}, T_i) - R)$, where $R$ is a fixed constant level of interest. (Actually, the contract is $Y_i = K\tau_i(L(T_{i-1}, T_i) - R)$ for some $K > 0$, but since $K$ is just a constant factor, we can set $K = 1$.)

a) Show that the value of the PFS swap at time $t$ equals

$$\Pi(t) = P(t, T_0) - P(t, T_n) - R \sum_{i=1}^{n} \tau_i P(t, T_i).$$

If we let $S_{T_n}(t)$ (with $T_n = (T_1, \ldots, T_n)$) be the fixed rate so that the PFS swap has zero value at time $t$, then we see that (this is obvious, you dont have to prove it)

$$S_{T_n}(t) = \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} \tau_i P(t, T_i)}.$$  

A cap is a sequence of caplets, so that at time $T_i$ it pays $\tau_i(L(T_{i-1}, T_i) - R)^+$, $i = 1, \ldots, n$.

b) Write down an expression for the value at time $t \leq T_0$ of a cap in terms of the values of the caplets. This is very easy, you must not do much.

A swaption is a $T_0$ contingent claim which at $T_0$ pays

$$Y = \left( \sum_{i=1}^{n} P(T_0, T_i) \tau_i(F(T_0, T_{i-1}, T_i) - R) \right)^+.$$  

c) Consider the $T_0$-contingent claim

$$Z = \sum_{i=1}^{n} P(T_0, T_i) \tau_i(F(T_0, T_{i-1}, T_i) - R)^+.$$  

Show that

$$\Pi_Y(t, T_0) \leq \Pi_Z(t, T_0).$$  

Referring to Exercise 1 above, discuss briefly how you can find $\Pi_Z(t, T_0)$.

d) Discuss briefly the problems that you encounter if you want to calculate $\Pi_Y(t, T_0)$, i.e. the value of the swaption.
Exercise 3

This exercise is a preparation for the market models to be introduced later. It is not difficult, and you can make extensive use of Exercise 2, HW 5.

Let $t < S < T$ and consider the caplet, which at delivery time $T$ pays

$$X = \tau (L(S, T) - R)^+,$$

where $\tau = \tau(S, T)$. We let the measure $P = \tilde{P}$, i.e. the risk neutral probability measure.

a) Find $\Pi_X(t, T)$ when the interest rate $r_t$ follows the Ho-Lee model

$$dr_t = \theta(t)dt + \sigma dW_t.$$

b) Same as a, but know we use the Hull-White model

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t.$$

Note that we can write $X$ as

$$X = \tau (F(S, S, T) - R)^+,$$

where $F(t, S, T)$ is the forward LIBOR rate. Assume that under the forward measure $P_T$,

$$dF(t, S, T) = F(t, S, T)\sigma_{S,T}dW^T_t, \quad t \leq S.$$

Then

$$\Pi_X(t, T) = P(t, T)\tau E^T[(F(S, S, T) - R)^+|\mathcal{F}_t].$$

Also assume that $\sigma_{S,T}$ is a constant.

c) Find $\Pi_X(t, T) = \text{Cpl}^{\text{Black}}(t, S, T; \sigma_{S,T}, R)$ as a function of the involved parameters. This is the market established Black formula for pricing of a caplet.

d) Is it possible to make appropriate choices of the parameters in either the Ho-Lee or the Hull-White model so that the valuation of a caplet becomes the same as when using Blakcs formula?