Exercise 1

Let $V_t$ be the price of a tradeable asset at time $t$. Using simple arbitrage arguments, show that the forward price for delivery of that asset at time $T > t$ equals

$$F_{V_T}(t, T) = \frac{V_t}{P(t, T)}.$$ 

Exercise 2

In this exercise you must show that the expected value of the simply compounded spot rate (LIBOR) equals the simply compounded forward rate, i.e. for $t < S < T$,

$$E^T[L(S, T)|\mathcal{F}_t] = F(t, S, T).$$

To do so, note that

$$F(t, S, T)P(t, T) = \frac{1}{\tau(S, T)}(P(t, S) - P(t, T)),$$

and the right-hand side here is obviously an $\mathcal{F}_t$-measurable tradable asset.

Exercise 3

In this exercise you must prove Lemma 0.1 in Lecture 4, which is: Let the stochastic variable $Y \sim N(\mu, \sigma^2)$. Then

a) $P(Y > k) = N\left(\frac{\mu - k}{\sigma}\right)$

b) $E\left[ e^{Y 1_{\{Y > k\}}} \right] = e^{\mu + \frac{1}{2} \sigma^2} N\left(\frac{\mu + \frac{1}{2} \sigma^2 - k}{\sigma}\right)$. 

**Exercise 4**

Consider the setup of Theorem 0.3, Lecture 4, where we price the European call option

\[ X = (P(T, T^*) - K)^+. \]

From Theorem 0.3, the only unknown factor is \( b^2(t, T, T^*) \).

a) Calculate \( b(t, T, T^*) \) for the Ho-Lee model

\[ dr_t = \theta(t) dt + \sigma d\tilde{W}_t. \]

b) Calculate \( b(t, T, T^*) \) for the Hull-White model

\[ dr_t = (\theta(t) - \alpha t) dt + \sigma d\tilde{W}_t. \]

In both cases, \( \tilde{W} \) is a Brownian motion under the risk neutral measure \( \tilde{P} \).

**Exercise 5**

In this exercise we will see (at least in theory) how we can use bonds to hedge a derivative. Let \( t < S < T \), and assume the value at delivery time \( S \) of the derivative \( X \) depends only on \( r_S \), i.e. \( X = \Phi(r_S) \). Assume the short rate dynamics follows

\[ dr_t = \mu(t, r_t) dt + \sigma(t, r_t) d\tilde{W}_t, \]

and that the bond dynamics follows

\[
\begin{align*}
    dP(t, S) &= r_t P(t, S) + v(t, S) P(t, S) d\tilde{W}_t, \\
    dP(t, T) &= r_t P(t, T) + v(t, T) P(t, T) d\tilde{W}_t.
\end{align*}
\]

We want to produce a self financing portfolio consisting of \( S \) bonds and \( T \) bonds, i.e.

\[ V_t = \phi_t P(t, S) + \eta_t P(t, T), \]

so that \( V_T = X \). We know that the value of that portfolio at time \( t \) must be

\[ V_t = \Pi_X(t, T) = \tilde{E} \left[ X e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \]

but that is not very important here.

Introduce the relative amounts invested in the two different bonds as \( u^S \) and \( u^T \), so that self financing yields

\[
\frac{dV_t}{V_t} = u^S_t \frac{dP(t, S)}{P(t, S)} + u^T_t \frac{dP(t, T)}{P(t, T)}
\]

with \( u^S_t + u^T_t = 1 \).
a) Explain why e.g.,

\[ \phi_t = \frac{u^S_t V_t}{P(t, S)}. \]

Assume now as in Lecture 2 that the value of derivative at time \( t \) can be written as \( \Pi_X(t, T) = F(t, r_t) \), where \( F \in C^{1,2} \).

b) Show that

\[
\phi_t = \frac{\sigma(t, r_t) F_x(t, r_t) - v(t, T) F(t, r_t)}{(v(t, S) - v(t, T)) P(t, S)},
\]

\[
\eta_t = \frac{F(t, r_t) - \phi_t P(t, S)}{P(t, T)}.
\]

Here some results from the second part of Lecture 2 may be useful.